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PRODUCTS OF STEINER'S QUASI-PROXIMITY SPACES

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ABSTRACT. E. F. Steiner introduced a quasi-proximity δ satisfying $A\ \delta\ B$ iff $\{x\}\ \delta\ B$ for some x of A. The purpose of this paper is to describe the Tychonoff product of topologies in terms of Steiner's quasi-proximities. Whenever (X_a, δ_a) is the Steiner quasi-proximity space, the product proximity on $X = \Pi X_a$ can be given, by using the concept of finite coverings, as the smallest proximity on X which makes each projection δ-continuous.

Introduction. E. F. Steiner [2] introduced a quasi-proximity δ satisfying $A\ \delta\ B$ iff $\{a\}\ \delta\ B$ for some a of A. This note is devoted to the study of a product proximity on $X = \Pi X_a$, where each (X_a, δ_a) is the above Steiner quasi-proximity space. As F. W. Stevenson [3] pointed out, there are three equivalent definitions of a product proximity. Especially, Császár and Leader defined a product proximity by using finite coverings [3]. Unfortunately, for Steiner's quasi-proximity, it seems difficult to us to define the product proximity in the same way as Császár and Leader. We must modify the definition of a product proximity in our case (Definition 2). We then show that the Tychonoff product topology can be induced on the cartesian product $X = \Pi X_a$ in terms of the quasi-proximity mentioned above.

The reader is referred to S. A. Naimpally and B. D. Warrack [1] for definitions not given here.

Preliminary definitions and lemmas.

Definition 1. A binary relation δ defined on the power set of X is called a *Steiner's* or *S-quasi-proximity* on X iff δ satisfies the axioms below.

(I) For every $A \subset X$, $A\ \overline{\delta}\ \phi$ ($\overline{\delta}$ means "not-δ").

(II) $A\ \delta\ B$ iff $\{a\}\ \delta\ B$ for some $a \in A$.

(III) $A\ \delta\ (B \cup C)$ iff $A\ \delta\ B$ or $A\ \delta\ C$.

(IV) For every $x \in X$, $\{x\}\ \delta\ \{x\}$.

(V) $A\ \overline{\delta}\ B$ implies that there exists a subset C such that $A\ \overline{\delta}\ C$ and $(X - C)\ \overline{\delta}\ B$.

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Remark 1. Clearly Axiom (II) is equivalent to Axiom (II′) below.

(II′) For an arbitrary index set Λ,

$$\left(\bigcup_{\lambda\in\Lambda} A_\lambda\right)\delta B \quad \text{iff } A_\mu \,\delta B \text{ for some } \mu\in\Lambda.$$

Furthermore, in the S-quasi-proximity we can replace Axiom (V) with Axiom (V′) below.

(V′) If $x \ll A$, then there exists a set B such that $x \ll B \ll A$. (In general, $P \ll Q$ means $P\,\bar{\delta}\,(X-Q)$ and Q is said to be a δ-neighborhood of P.)

In fact, it is easily seen that Axiom (V) implies Axiom (V′). Conversely we show that Axiom (V) follows from Axioms (I)–(IV) and (V′). Suppose $A\,\bar{\delta}\,B$. By Axiom (II), $\{x\}\,\bar{\delta}\,B$, i.e. $x \ll X-B$ for each $x\in A$. Then it follows from Axiom (V′) that there is a set C_x such that $x \ll C_x \ll X-B$ for each $x\in A$. Since $\{x\}\,\bar{\delta}\,(X-C_x)$ for each $x\in A$,

$$\{x\}\,\bar{\delta}\left(X-\bigcup_{x\in A} C_x\right) \quad \text{by Axiom (III).}$$

Setting $\bigcup_{x\in A} C_x = C$, we obtain $A\,\bar{\delta}\,(X-C)$ by Axiom (II). On the other hand, since $C_x\,\bar{\delta}\,B$ for each $x\in A$, we have $C\,\bar{\delta}\,B$ by Axiom (II′). Thus Axiom (V) surely holds.

Let (X,δ) be an S-quasi-proximity space. For every $A\subset X$, we set $c(A)=\{x\colon \{x\}\,\delta\,A\}$. Then the operator c is a topological closure operator and so X is a topological space [2]. This topological space is denoted by (X, c) and the topology induced by δ is denoted by $\tau(\delta)$. If, on a set X, there is a topology τ and a proximity δ such that $\tau=\tau(\delta)$, then τ and δ are said to be *compatible.*

The proof of the following is trivial.

Lemma 1. (1) *If* $A\,\delta\,B$ *and* $B\subset C$, *then* $A\,\delta\,C$.
(2) *If* $A\,\delta\,B$ *and* $A\subset C$, *then* $C\,\delta\,B$.
(3) *If* $A\,\bar{\delta}\,B$, *then* $A\cap B=\emptyset$.

Lemma 2. *For subsets* A *and* B *of an* S*-quasi-proximity space* (X, c),

$$A\,\delta\,B \quad \textit{iff}\ A\cap c(B)\neq\emptyset \quad \textit{iff}\ A\,\delta\,c(B).$$

Proof. This follows readily from Axiom (II).

The following is a direct result of Lemma 2.

Lemma 3. *Every topological space* (X,τ) *with the topology* τ *has a* *compatible S-quasi-proximity δ defined by*

$$A\,\delta\,B \quad \textit{iff} \quad A \cap \overline{B} \neq \emptyset,$$

where $\overline{B}$ denotes the τ-closure of B.

The following lemma shows that in S-quasi-proximity spaces a δ-continuous mapping and a continuous mapping are equivalent.

Lemma 4. *Let f be a mapping of an S-quasi-proximity space (X, δ_1) into an S-quasi-proximity space (Y, δ_2). Then f is δ-continuous if and only if it is a continuous mapping of the topological space $(X, \tau(\delta_1))$ into the topological space $(Y, \tau(\delta_2))$.*

Proof. Suppose that f is δ-continuous and that x is any point of $c_1(A)$. Then $\{x\}\,\delta_1\,A$, which implies $f(x)\,\delta_2\,f(A)$. It follows that $f(x) \in c_2\{f(A)\}$ and so $f\{c_1(A)\} \subset c_2\{f(A)\}$. ($c_1$ and c_2 denote the closure operators in (X, δ_1) and (Y, δ_2) respectively.) Conversely let f be continuous and let $A\,\delta_1\,B$. Since, by Lemma 2 $A \cap c_1(B) \neq \emptyset$, it follows that $f(A) \cap f\{c_1(B)\} \neq \emptyset$. From the continuity of f, we obtain that $f(A) \cap c_2\{f(B)\} \neq \emptyset$. This implies $f(A)\,\delta_2\,f(B)$, so that f is δ-continuous. Q. E. D.

Proximity products. In the present section we attempt to obtain a direct construction of an S-quasi-proximity product space by a *proximal* approach. As we stated in the introduction, we modify the definition of Császár and Leader for the product proximity.

Definition 2. Let $\{(X_a, \delta_a): a \in \Lambda\}$ be an arbitrary family of S-quasi-proximity spaces. Let $X = \Pi_{a\in\Lambda} X_a$ denote the cartesian product of these spaces. A binary relation δ on the power set of X is defined as follows:

Let A and B be subsets of X. Define $A\,\delta\,B$ iff there is a point $x_0 \in A$ such that, for any finite covering $\{B_i: i = 1, 2, \ldots, n\}$ of B, there exists a set B_i satisfying $P_a[x_0]\,\delta_a\,P_a[B_i]$ for each $a \in \Lambda$, where each P_a denotes the projection from X to X_a.

Remark 2. Leader [3] defined a product proximity as follows: For $A, B \subset X$, $A\,\delta\,B$ iff for any finite coverings $\{A_i: i = 1, 2, \ldots, m\}$ and $\{B_j: j = 1, 2, \ldots, n\}$ of A and B respectively, there is an A_i and a B_j such that $P_a[A_i]\,\delta_a\,P_a[B_j]$ for each $a \in \Lambda$. But in order to prove that δ satisfies Axiom (II), it seems difficult to use Leader's definition for the S-quasi-proximity.

Lemma 5. *Let each (X_a, δ_a) be an S-quasi-proximity space and let A*

and B be subsets of $X = \Pi X_a$. Then $A \delta B$ implies $P_a[A] \delta_a P_a[B]$ for each $a \in \Lambda$.

Proof. Suppose $A \delta B$. Since $\{B\}$ itself is a finite covering of B, there is a point x_0 of A such that $P_a[x_0] \delta_a P_a[B]$ for each $a \in \Lambda$. Applying Axiom (II) to each δ_a, we have $P_a[A] \delta_a P_a[B]$ for each $a \in \Lambda$. Q. E. D.

It follows from Lemma 5 that each projection P_a is δ-continuous and hence it is also continuous by Lemma 4 if X becomes an S-quasi-proximity space. Now we prove the main theorem.

Theorem 1. *The binary relation δ given by Definition 2 is an S-quasi-proximity on the cartesian product X. This space (X, δ) is said to be an S-quasi-proximity product space.*

Proof. It suffices to show that δ satisfies Axioms (I)–(IV) of Definition 1 and Axiom (V′) of Remark 1. It is easy to see that δ satisfies Axiom (I).

Axiom (II): Suppose $A \delta B$. If $x_0 \in A$ fulfils the condition in Definition 2, then clearly $\{x_0\} \delta B$.

Conversely suppose that $\{x_0\} \delta B$ for some x_0 of A. If $\{B_i: i = 1, 2, \ldots, n\}$ is any finite covering of B, then there is a set B_i such that $P_a[x_0] \delta_a P_a[B_i]$ for each $a \in \Lambda$. By Definition 2, this means $A \delta B$.

Axiom (III): Suppose $A \delta B$ and let $x_0 \in A$ satisfy the condition in Definition 2. If $\{D_i: i = 1, 2, \ldots, n\}$ is any finite covering of $B \cup C$, then it is a covering of B as well; hence there is an i such that $P_a[x_0] \delta_a P_a[D_i]$ for each $a \in \Lambda$. Thus $A \delta (B \cup C)$.

Conversely suppose $A \bar{\delta} B$ and $A \bar{\delta} C$. Then for any given $x \in A$, there are finite coverings $\{D_i: i = 1, 2, \ldots, n\}$ and $\{D_j: j = n + 1, \ldots, n + p\}$ of B and C respectively such that

$$P_a[x] \bar{\delta}_a P_a[D_i] \quad \text{for } a = t_i \in \Lambda,$$

$$P_a[x] \bar{\delta}_a P_a[D_j] \quad \text{for } a = s_j \in \Lambda,$$

where $i = 1, 2, \ldots, n$ and $j = n + 1, \ldots, n + p$. Since $\{D_k: k = 1, 2, \ldots, n + p\}$ is a covering of $B \cup C$, we conclude that $A \bar{\delta} (B \cup C)$.

Axiom (IV): Let x be a point of X and let A be any set such that $x \in A$. Since $P_a[x] \in P_a[A]$ for each $a \in \Lambda$, by Lemma 1(3) we have $P_a[x] \delta_a P_a[A]$ for each $a \in \Lambda$. Thus $\{x\} \delta \{x\}$.

Axiom (V′): Let $\{x\}$ and A be subsets of X such that $x \ll A$, that is, $\{x\} \bar{\delta} (X - A)$. Then there is a finite covering $\{A_i: i = 1, 2, \ldots, n\}$ of $(X - A)$ such that $P_a[x] \bar{\delta}_a P_a[A_i]$ for some $a = t_i \in \Lambda$, where $i = 1, 2, \ldots, n$.

Equivalently $P_a[x] \ll X_a - P_a[A_i]$. Since each δ_a satisfies Axiom (V′), there exist G_i $(i = 1, 2, \ldots, n)$ such that

(1) $$P_a[x] \ll G_i \ll X_a - P_a[A_i] \quad \text{for } a = t_i \in \Lambda.$$

From the first half of (1), we have

(2) $$P_a[x]\,\overline{\delta}_a(X_a - G_i).$$

Now we set

$$K_i = P_a^{-1}[X_a - G_i] = X - P_a^{-1}[G_i]$$

and set $K = \bigcup_{i=1}^{n} K_i$. It follows from (2) that

$$P_a[x]\,\overline{\delta}_a\,P_a[K_i] \quad \text{for } a = t_i \in \Lambda, \qquad i = 1, 2, \ldots, n.$$

Since $\{K_i: i = 1, 2, \ldots, n\}$ is a finite covering of K, we obtain $\{x\}\,\overline{\delta}\,K$. This implies

(3) $$x \ll X - K.$$

Next, from the second half of (1), we have

(4) $$G_i\,\overline{\delta}_a\,P_a[A_i] \quad \text{for some } a = t_i, \qquad i = 1, 2, \ldots, n.$$

On the other hand, since

$$X - K = \bigcap_{j=1}^{n} P_a^{-1}[G_j] \qquad (a = t_j),$$

it follows that

$$P_a[X - K] = P_a\left\{\bigcap_{j=1}^{n} P_{t_j}^{-1}[G_j]\right\} \subset G_i \quad \text{for } a = t_i.$$

Hence for every point y of $X - K$,

$$P_a[y] \in G_i \qquad (a = t_i;\ i = 1, 2, \ldots, n).$$

By (4) and Lemma 1(2), we have therefore $P_a[y]\,\overline{\delta}_a\,P_a[A_i]$ for every y of $X - K$, where $a = t_i$; $i = 1, 2, \ldots, n$. Because $\{A_i: i = 1, 2, \ldots, n\}$ is a finite covering of $(X - A)$, we get that

(5) $$(X - K)\,\overline{\delta}\,(X - A), \quad \text{that is,} \quad X - K \ll A.$$

Relations (3) and (5) together show that δ satisfies Axiom (V′). This completes the proof.

In view of Lemma 4, the following theorem shows that the Tychonoff product topology can be induced on an S-quasi-proximity product space $(X, \tau(\delta))$.

Theorem 2. *The S-quasi-proximity δ on X given by Definition 2 is the smallest S-quasi-proximity for which each projection P_a is δ-continuous.*

Proof. Let β be an arbitrary S-quasi-proximity on X such that each projection P_a is a δ-continuous mapping of (X, β) into (X_a, δ_a). Then we must show that $A \beta B$ implies $A \delta B$ for $A, B \subset X$. By Axiom (II), there is a point x_0 of A such that $\{x_0\} \beta B$. Given any finite covering $\{B_i: i = 1, 2, \ldots, n\}$ of B, we can choose a set B_i such that $\{x_0\} \beta B_i$ by Axiom (III). Since each P_a is δ-continuous, $P_a[x_0] \delta_a P_a[B_i]$ for each $a \in \Lambda$. Because of Definition 2, we can conclude $A \delta B$. Q. E. D.

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