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THE MONAD SYSTEM OF THE FINEST COMPATIBLE UNIFORM STRUCTURE

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ABSTRACT. The methods of nonstandard analysis are used to study the finest uniform structure compatible with the topology on a given completely regular, Hausdorff space.

Let (X, τ) be a completely regular, Hausdorff space and let $\mathfrak{F}$ be the finest uniform structure on X which is compatible with τ. If ${}^*X$ is the set corresponding to X in some enlargement, then $\mathfrak{F}$ corresponds to a partition $\{\mu_{\mathfrak{F}}(p): p \in {}^*X\}$ of ${}^*X$ in a familiar way [5], [6]. This partition may be described by

$$q \in \mu_{\mathfrak{F}}(p) \Longleftrightarrow {}^*d(p, q) \text{ is infinitesimal for every continuous pseudometric } d \text{ on } (X, \tau).$$

The results in this paper concern the structure of the monad system $\mu_{\mathfrak{F}}$, especially as it is related to the set $C(X)$ of continuous, real-valued functions on (X, τ). Also, it is proved that $\mu_{\mathfrak{F}}$ is identical to the μ-monad system constructed in a quite different way by Wattenberg [8] and some consequences of this fact are discussed.

In general we adopt in this paper the framework for nonstandard analysis which is described in Luxemburg's important paper [5]. Throughout this paper $\mathfrak{M}$ will denote a higher order set-theoretical structure and ${}^*\mathfrak{M}$ will denote an enlargement of $\mathfrak{M}$ which is $\aleph_0$-enlarging in the sense of [4]. Given a uniform space $(X, \mathfrak{U})$ in $\mathfrak{M}$, we let $\overset{\mathfrak{U}}{=}$ be the equivalence relation on ${}^*X$ which corresponds to $\mathfrak{U}$; that is

$$p \overset{\mathfrak{U}}{=} q \Longleftrightarrow (p, q) \in {}^*V \text{ for all } V \in \mathfrak{U}.$$

For each $p \in {}^*X$ the $\mathfrak{U}$-monad of p, which is denoted by $\mu_{\mathfrak{U}}(p)$, is the equivalence class of p under $\overset{\mathfrak{U}}{=}$. A discussion of how the monads $\mu_{\mathfrak{U}}(p)$ form a basis for the nonstandard theory of uniform spaces may be found in [5] or [6].

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A uniform structure $\tilde{\mathfrak{U}}$ is generated on ${}^*X$ by the collection $\{{}^*V\colon V \in \mathfrak{U}\}$ of subsets of ${}^*X \times {}^*X$, as described in [3]. For each $p \in {}^*X$, the monad $\mu_{\mathfrak{U}}(p)$ is then equal to the intersection of all $\tilde{\mathfrak{U}}$-neighborhoods of p. Therefore, $p \overset{\mathfrak{U}}{=} q$ if and only if p and q have exactly the same filter of $\tilde{\mathfrak{U}}$-neighborhoods. Let $X_0 = \{\mu_{\mathfrak{U}}(p)\colon p \in {}^*X\}$ and let $\mathfrak{U}_0$ be the quotient uniformity induced on X_0 by $\tilde{\mathfrak{U}}$. Denote the quotient map from $({}^*X, \tilde{\mathfrak{U}})$ onto $(X_0, \mathfrak{U}_0)$ by $\pi_{\mathfrak{U}}$.

An element p of ${}^*X$ is called $\mathfrak{U}$-prenearstandard [5] if for each $V \in \mathfrak{U}$ there exists $x \in X$ with $(p, {}^*x) \in {}^*V$. The set of $\mathfrak{U}$-prenearstandard points is denoted by $\mathrm{pns}_{\mathfrak{U}}$. Note that $\mathrm{pns}_{\mathfrak{U}}$ is just the $\tilde{\mathfrak{U}}$-closure in ${}^*X$ of the set $\{{}^*x\colon x \in X\}$ of standard points in ${}^*X$. It can be shown that $\pi_{\mathfrak{U}}(\mathrm{pns}_{\mathfrak{U}})$ is $\mathfrak{U}_0$-complete so that if $(X, \mathfrak{U})$ is Hausdorff, then the subspace $\pi_{\mathfrak{U}}(\mathrm{pns}_{\mathfrak{U}})$ of $(X_0, \mathfrak{U}_0)$ is the completion of $(X, \mathfrak{U})$.

Our purpose in assuming that ${}^*\mathfrak{M}$ is $\aleph_0$-enlarging is to obtain a local description of the $\tilde{\mathfrak{U}}$-topology on ${}^*X$ in terms of the $\mathfrak{U}$-monads, as given in Theorem 1. Recall that ${}^*\mathfrak{M}$ is $\aleph_0$-enlarging if and only if it has the following property described in [5]: if Y is a set in $\mathfrak{M}$, $\mathcal{G}$ is a filter on Y and A is an internal subset of ${}^*Y$, then $A \cap \mu(\mathcal{G}) \neq \emptyset$ if and only if $A \cap {}^*W \neq \emptyset$ for every $W \in \mathcal{G}$. (Here $\mu(\mathcal{G})$ is the filter monad of $\mathcal{G}$ defined by $\mu(\mathcal{G}) = \bigcap\{{}^*W\colon W \in \mathcal{G}\}$.) Recall also that for each set X there exists ${}^*\mathfrak{M}$ which is an $\aleph_0$-enlarging extension of some $\mathfrak{M}$ containing X. (For example, the limit of a sequence of successive enlargements, the first of which has X as an element.)

Theorem 1. *Let* $(X, \mathfrak{U})$ *be a uniform space in* $\mathfrak{M}$.

(i) *If* $A \subseteq {}^*X$ *is internal, then the interior of* A *in the* $\tilde{\mathfrak{U}}$*-topology is* $\{p\colon \mu_{\mathfrak{U}}(p) \subseteq A\}$.

(ii) *For* $p \in {}^*X$, *a basis for the* $\tilde{\mathfrak{U}}$*-neighborhood filter at* p *consists of those internal subsets of* ${}^*X$ *which contain* $\mu_{\mathfrak{U}}(p)$.

Proof. (i) If p is in the $\tilde{\mathfrak{U}}$-interior of A, then ${}^*V(p) \subseteq A$ for some $V \in \mathfrak{U}$ and therefore $\mu_{\mathfrak{U}}(p) \subseteq A$. If p is not in the interior of A, then ${}^*V(p)$ intersects ${}^*X \sim A$ for every $V \in \mathfrak{U}$. Since ${}^*\mathfrak{M}$ is $\aleph_0$-enlarging it follows that the filter monad of $\mathfrak{U}$, which is equal to the equivalence relation $\overset{\mathfrak{U}}{=}$, intersects $({}^*X \sim A) \times \{p\}$. Therefore $\mu_{\mathfrak{U}}(p) \nsubseteq A$ if p is not in the interior of A.

(ii) This is an immediate consequence of (i) and the fact that the $\tilde{\mathfrak{U}}$-neighborhood filter at p has a basis $\{{}^*V(p)\colon V \in \mathfrak{U}\}$ of internal sets.

The observations in Theorem 1 are useful in the following type of setting: $\mathfrak{U}$ and $\mathfrak{V}$ be two uniform structures on X and suppose that S is a subset of ${}^*X$ such that $\mu_{\mathfrak{U}}(p) \subseteq \mu_{\mathfrak{V}}(p)$ for every $p \in S$. Then there is a natural function ϕ from $\pi_{\mathfrak{U}}(S)$ onto $\pi_{\mathfrak{V}}(S)$ which takes $\pi_{\mathfrak{U}}(p)$ to $\pi_{\mathfrak{V}}(p)$ for each $p \in S$. Theorem 1 implies that the $\tilde{\mathfrak{U}}$-topology is finer than the $\tilde{\mathfrak{V}}$-topology when restricted to S. It follows that ϕ is continuous (relative to the $\mathfrak{U}_0$-topology on $\pi_{\mathfrak{U}}(S)$ and the $\mathfrak{V}_0$-topology on $\pi_{\mathfrak{V}}(S)$). In particular, if $\mu_{\mathfrak{U}}(p) = \mu_{\mathfrak{V}}(p)$ for every $p \in S$, then ϕ is a homeomorphism.

Now let (X, τ) be a completely regular, Hausdorff space in $\mathfrak{M}$ and let $\mathfrak{F}$ be the finest compatible uniform structure on (X, τ). In the next result, the difficult part of which is a consequence of Shirota's theorem [2, Theorem 15.21], the relation between $\mathfrak{F}$ and $C(X)$ is explored in terms of nonstandard analysis.

Theorem 2. *For each* $p \in X^*$

$$(1) \qquad \mu_{\mathfrak{F}}(p) \subseteq \{q\colon {}^*f(p) =_1 {}^*f(q) \text{ for all } f \in C(X)\}.$$

If p *is* $\mathfrak{F}$*-prenearstandard, then equality holds in* (1). *Moreover, if* (X, τ) *has no closed, discrete subspace of measurable cardinality, then*

$$(2) \qquad \mathrm{pns}_{\mathfrak{F}} = \{p\colon {}^*f(p) \text{ is finite for all } f \in C(X)\}.$$

Proof. For each $f \in C(X)$ the function $|f(x) - f(y)|$ is a continuous pseudometric on (X, τ). Therefore $p \overset{\mathfrak{F}}{=} q$ implies ${}^*f(p) =_1 {}^*f(q)$ for any $p, q \in X$ and $f \in C(X)$. This shows that (1) holds in general.

To prove that equality holds in (1) for prenearstandard points, let $p \in \mathrm{pns}_{\mathfrak{F}}$. If $q \notin \mu_{\mathfrak{F}}(p)$, then for some standard $\delta > 0$ and some continuous pseudometric d on (X, τ), ${}^*d(p, q) > \delta$. Since $p \in \mathrm{pns}_{\mathfrak{F}}$ there exists $x \in X$ which satisfies ${}^*d(p, {}^*x) < \delta/3$. The function $f(y) = d(y, x)$ is in $C(X)$, but ${}^*f(p)$ cannot be infinitely close to ${}^*f(q)$ (since otherwise ${}^*d(q, {}^*x) < \delta/2$ and hence ${}^*d(p, q) < \delta$). Thus q is also not in the set on the right side of (1).

Now let $\mathfrak{U}$ be the uniform structure on X generated by the pseudometrics $|f(x) - f(y)|$ for $f \in C(X)$. Evidently $\mathfrak{U} \subseteq \mathfrak{F}$ and therefore $\mathrm{pns}_{\mathfrak{F}} \subseteq \mathrm{pns}_{\mathfrak{U}}$. Moreover, $\mathrm{pns}_{\mathfrak{U}}$ is equal to the right side of (2) by [5, Theorem 3.15.5]. By the remarks above, a homeomorphism ϕ of $\pi_{\mathfrak{F}}(\mathrm{pns}_{\mathfrak{F}})$ into $\pi_{\mathfrak{U}}(\mathrm{pns}_{\mathfrak{U}})$ may be defined by setting $\phi(\pi_{\mathfrak{F}}(p))$ equal to $\pi_{\mathfrak{U}}(p)$ for $p \in \mathrm{pns}_{\mathfrak{F}}$. Now $\pi_{\mathfrak{F}}(\mathrm{pns}_{\mathfrak{F}})$ is the completion of $(X, \mathfrak{F})$ and $\pi_{\mathfrak{U}}(\mathrm{pns}_{\mathfrak{U}})$ is the completion of $(X, \mathfrak{U})$. In case the stated assumption holds, then Shirota's theorem [2, Theorem 15.21] implies that both of these completions are real compact. Also they both contain X densely and ϕ is the identity map on X. Since every function in $C(X)$ extends to a continuous function on $\pi_{\mathfrak{U}}(\mathrm{pns}_{\mathfrak{U}})$ it follows that ϕ maps $\pi_{\mathfrak{F}}(\mathrm{pns}_{\mathfrak{F}})$ onto $\pi_{\mathfrak{U}}(\mathrm{pns}_{\mathfrak{U}})$. The first part of the theorem then implies (2).

Next we show that the uniform structure $\mathfrak{F}$ is closely related to the monad systems introduced by Wattenberg in [7] and [8]. The most interesting of these systems, which we will call the μ-monad system, is constructed as follows [8, Definition 2.9]: given a completely regular, Hausdorff space (X, τ) in $\mathfrak{M}$, let λ be a cardinal number which is greater than $2^{\aleph_0}$ and greater than the cardinality of X. Let S be a set of cardinality λ and let $\mathbf{R}^{\lambda}$ be the set of real-valued functions α on S whose support $\{s: \alpha(s) \neq 0\}$ is finite. Equip $\mathbf{R}^{\lambda}$ with the metric d_{λ} defined by

$$d_{\lambda}(\alpha, \beta) = \sup\{|\alpha(s) - \beta(s)|: s \in S\}.$$

The μ-monad for (X, τ) consists of the family $\{\mu(p): p \in {}^{*}X\}$ of subsets of ${}^{*}X$ defined by

$$\mu(p) = \{q: \text{for each continuous function}$$
$$f: (X, \tau) \to (\mathbf{R}^{\lambda}, d_{\lambda}),\ {}^{*}d_{\lambda}({}^{*}fp, {}^{*}fq) =_1 0\}.$$

The significance of the μ-monad system lies in these facts, proved in [8]:

(i) $\{\mu(p): p \in {}^{*}X\}$ is a partition of ${}^{*}X$ which agrees with the τ-monad partition on τ-nearstandard points;

(ii) if $f: (X_1, \tau_1) \to (X_2, \tau_2)$ is a continuous function between completely regular, Hausdorff spaces in $\mathfrak{M}$, then for each $p \in {}^{*}X_1$, ${}^{*}f(\mu(p)) \subseteq \mu({}^{*}f(p))$. Moreover, the μ-monad system is the same as the metric monad system on metric spaces and is the same as the covering monad system on normal spaces [8, Theorem 2.10], both of which have topologically natural definitions.

Our next result is that the μ-monad on ${}^{*}X$ is identical to the monad $\mu_{\mathfrak{F}}$ given by the finest compatible uniform structure $\mathfrak{F}$ on (X, τ). This theorem not only makes a connection between Wattenberg's work and the theory of uniform spaces, but is also leads (via his work) to useful descriptions of the $\mu_{\mathfrak{F}}$-monads in certain special situations.

Theorem 3. *If (X, τ) is a completely regular, Hausdorff space in $\mathfrak{M}$ and $\mathfrak{F}$ is the finest compatible uniform structure on (X, τ), then $\mu(p) = \mu_{\mathfrak{F}}(p)$ for all $p \in {}^{*}X$.*

Proof. Let the μ-monad for (X, τ) be constructed as described above. The definition of $\mu(p)$ for $p \in {}^*X$ makes it evident that there is a filter $\mathfrak{U}$ on $X \times X$ such that the filter monad of $\mathfrak{U}$ equals $\{(p, q)\colon q \in \mu(p)\}$. Since this set is an equivalence relation on ${}^*X$, it follows that $\mathfrak{U}$ is a uniform structure on X [5, Theorem 3.9.1]. In addition, $\mu(p) = \mu_{\mathfrak{U}}(p)$ for all $p \in {}^*X$. By (i) above, for each $x \in X$

$$\mu_{\mathfrak{U}}(x) = \{p\colon p \text{ is } \tau\text{-nearstandard to } x\} = \mu_\tau(x).$$

This implies that the uniform structure $\mathfrak{U}$ is compatible with τ and hence $\mathfrak{U} \subseteq \mathfrak{F}$.

Now $\mathfrak{F}$ may be described as the coarsest uniform structure on X for which every continuous function from (X, τ) into a metric space is actually uniformly continuous. Let (Y, d) be a metric space and $f\colon (X, \tau) \to (Y, d)$ a continuous function. To prove that f is uniformly continuous relative to $\mathfrak{U}$ it suffices to show

$$(3) \qquad {}^*f(\mu(p)) \subseteq \{q\colon {}^*d(q, {}^*fp) =_1 0\}$$

for each $p \in {}^*X$. Property (ii) of the μ-monad system implies that ${}^*f(\mu(p)) \subseteq \mu({}^*f(p))$. But the μ-monad for (Y, d) is the same as the metric monad [8, Theorem 2.10] which is finer than the d-monad. That is

$$\mu({}^*f(p)) \subseteq \{q\colon {}^*d(q, {}^*fp) =_1 0\}$$

which yields (3). That is, each continuous function from (X, τ) into a metric space is uniformly continuous relative to $\mathfrak{U}$, and therefore $\mathfrak{F} \subseteq \mathfrak{U}$. This proves $\mathfrak{U} = \mathfrak{F}$ so that $\mu(p) = \mu_{\mathfrak{U}}(p) = \mu_{\mathfrak{F}}(p)$ for every $p \in {}^*X$.

Corollary 1. *If (X, d) is a metric space in $\mathfrak{M}$ and $\mathfrak{F}$ is the finest uniform structure compatible with the d-topology on X, then for $p \in {}^*X$*

$$\mu_{\mathfrak{F}}(p) = \{q\colon {}^*d(p, q) < {}^*f(p) \text{ for every positive function } f \text{ in } C(X)\}.$$

Proof. Theorem 3 and [8, Theorem 2.10].

The description of the $\mathfrak{F}$-monads given in Corollary 1 enables us to give a simple proof of a result for subspaces of the Euclidean spaces which is contained in the paper [1] of Corson and Isbell.

Corollary 2. *If (X, τ) is homeomorphic to a subspace of $\mathbf{R}^n$ for some $n \geq 1$ and $\mathfrak{F}$ is the finest compatible uniform structure on (X, τ), then $\mathfrak{F}$ is the uniform structure on X generated by the pseudometrics $|f(x) - f(y)|$ for $f \in C(X)$. That is, $\mathfrak{F}$ is the unique compatible uniform structure on*

(X, τ) which makes every function in $C(X)$ uniformly continuous.

Proof. We assume $X \subseteq \mathbf{R}^n$ and let τ be the subspace topology so τ is defined by the metric

$$d(x, y) = \max\{|x_i - y_i|: 1 \leq i \leq n\}$$

where $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_n)$. By Corollary 1 it suffices to show that if ${}^*f(p) =_1 {}^*f(q)$ for every $f \in C(X)$, then ${}^*d(p, q) < {}^*g(p)$ for every positive $g \in C(X)$. To prove this, let p, q satisfy the first condition and let $g \in C(X)$ be positive. We will show ${}^*d(p, q) < {}^*g(p)$, and it may be assumed that g is bounded on X. Then

$$1 =_1 {}^*g(q)/{}^*g(p) \tag{4}$$

since $1/g(x)$ is in $C(X)$ and ${}^*g(q)$ is finite. Let $p = (p_1, \ldots, p_n)$ and $q = (q_1, \ldots, q_n)$. Since $x_i/g(x)$ is in $C(X)$, it follows from (4) that

$$p_i/{}^*g(p) =_1 q_i/{}^*g(p).$$

This shows that $|p_i - q_i| < {}^*g(p)$ for $1 \leq i \leq n$ and therefore ${}^*d(p, q) < {}^*g(p)$, completing the proof.

Now let (X, τ) be an arbitrary completely regular, Hausdorff space in $\mathfrak{M}$ and let $\mathfrak{F}$ be the finest compatible uniform structure on (X, τ). Two elements p, q of ${}^*X$ are in the same $\mathfrak{F}$-galaxy [3] if for each $V \in \mathfrak{F}$ there is a finite sequence $p_0, \ldots, p_k$ in ${}^*X$ such that $p_0 = p$, $p_k = q$ and $(p_i, p_{i+1}) \in {}^*V$ for each $i = 0, \ldots, k-1$. In particular this implies that ${}^*d(p, q)$ is finite for every continuous pseudometric d on (X, τ). Therefore, the set $A = \{p: {}^*f(p)$ is finite for all $f \in C(X)\}$ is a union of $\mathfrak{F}$-galaxies, as is ${}^*X \sim A$.

If there are no measurable cardinals, then Theorem 2 shows that $\pi_{\mathfrak{F}}(A)$ is the completion of $(X, \mathfrak{F})$. Therefore the galaxy structure for points in A reflects in an immediate way the structure of (X, τ). The next result shows that the galaxy structure for points outside A is trivial. This has the immediate consequence that any connected subset of $X_0 = \pi_{\mathfrak{F}}({}^*X)$ with at least two elements must be contained in $\pi_{\mathfrak{F}}(A)$. That is, $\pi_{\mathfrak{F}}(A)$ is open and closed in X_0 and $X_0 \sim \pi_{\mathfrak{F}}(A)$ is totally disconnected (in the $\mathfrak{F}_0$-topology on X_0).

Theorem 4. *Let (X, τ) be a completely regular, Hausdorff space in $\mathfrak{M}$ and let $\mathfrak{F}$ be the finest compatible uniform structure on (X, τ). If $p \in {}^*X$ and there exists $f \in C(X)$ such that ${}^*f(p)$ is infinite, then the $\mathfrak{F}$-galaxy containing p is equal to the monad $\mu_{\mathfrak{F}}(p)$.*

Proof. Suppose $p \in {}^*X$ and $f \in C(X)$ and assume that ${}^*f(p)$ is infinite. Let q be an element of the same $\mathfrak{F}$-galaxy as p, so that ${}^*d(p, q)$ is finite for every continuous pseudometric d on (X, τ) [3, Theorem 4.2].

We may assume $f \geq 1$; let $g = 1/f \in C(X)$ so $0 < g \leq 1$ and ${}^*g(p) =_1 0$. Given a continuous pseudometric d, define

$$d'(x, y) = d(x, y) + |g(x) - g(y)|$$

and

$$V_n = \{(x, y): d'(x, y) < n^{-1} \cdot g(x)\}$$

for $n \geq 1$. Since d' is a continuous pseudometric on (X, τ), the sets V_n are open neighborhoods of the diagonal in $X \times X$. We will show that $V_{n+1} \subseteq (V_n)^{-1}$ and $(V_{4n+1})^2 \subseteq V_n$ for each $n \geq 1$. Suppose $d'(x, y) < (n+1)^{-1} \cdot g(x)$. Then $|g(x) - g(y)| < (n+1)^{-1} \cdot g(x)$ so that $(n+1)^{-1} \cdot g(x) < n^{-1}g(y)$ and $g(y) < (n+2)(n+1)^{-1} \cdot g(x)$. It follows that $d'(x, y) < n^{-1}g(y)$, which proves $V_{n+1} \subseteq (V_n)^{-1}$. If also $d'(y, z) < (n+1)^{-1} \cdot g(y)$, then

$$d'(x, z) < ((n+1)^{-1} + n^{-1})g(y) < 4n^{-1}g(x).$$

This proves $(V_{4n+1})^2 \subseteq V_n$. It follows from these facts that $\{V_n: n \geq 1\}$ generates a uniform structure on X which defines a topology coarser than τ. Therefore $\{V_n: n \geq 1\}$ is contained in $\mathfrak{F}$.

Now since V_1 is in $\mathfrak{F}$ and since p and q are in the same $\mathfrak{F}$-galaxy, there is a finite sequence $p_0, \ldots, p_k$ in ${}^*X$ such that $p_0 = p$, $p_k = q$ and for each $i = 0, \ldots, k-1$

$${}^*d(p_i, p_{i+1}) + |{}^*g(p_i) - {}^*g(p_{i+1})| < {}^*g(p_i).$$

An inductive argument shows that ${}^*g(p_i) =_1 0$ for each $i = 0, \ldots, k-1$. Using the triangle inequality for ${}^*d$ yields

$${}^*d(p, q) < \sum_{i=0}^{k-1} {}^*g(p_i) =_1 0.$$

That is, ${}^*d(p, q) =_1 0$ for every continuous pseudometric d on (X, τ). Therefore $q \in \mu_{\mathfrak{F}}(p)$ and the proof is complete.

REFERENCES

1. H. H. Corson and J. R. Isbell, *Euclidean covers of topological spaces*, Quart. J. Math. Oxford Ser. (2) 11 (1960), 34–42. MR 23 #A2194.

2. Leonard Gillman and Meyer Jerison, *Rings of continuous functions*, University Ser. in Higher Math., Van Nostrand, Princeton, N. J., 1960. MR 22 #6994.

3. C. Ward Henson, *The nonstandard hulls of a uniform space*, Pacific J. Math. 43 (1972), 115–137. MR 47 #2559.

4. C. Ward Henson and L. C. Moore, Jr., *The nonstandard theory of topological vector spaces*, Trans. Amer. Math. Soc. 172 (1972), 405–435. MR 46 #7836.

5. W. A. J. Luxemburg, *A general theory of monads*, Applications of Model Theory to Algebra, Analysis and Probability, Holt, Rinehart and Winston, New York, 1969, pp. 18–86. MR 39 #6244.

6. M. Machover and J. Hirschfeld, *Lectures on nonstandard analysis*, Lecture Notes in Math., Vol. 94, Springer-Verlag, Berlin, 1969. MR 40 #2531.

7. Frank Wattenberg, *Nonstandard topology and extensions of monad systems to infinite points*, J. Symbolic Logic 36 (1971), 463–476. MR 45 #2642.

8. ———, *Monads of infinite points and finite product spaces*, Trans. Amer. Math. Soc. 176 (1973), 351–368. MR 47 #1020.

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