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ON MATRIX APPROXIMATION[1]

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ABSTRACT. In this paper we give an algebraic characterization of the best approximants to a given matrix A from a real line spanned by a matrix B. The distance $\|A - \alpha B\|$ is taken to be the spectral norm of $A - \alpha B$.

1. Introduction. Let A and B be $m \times n$-complex-valued nonzero matrices. Assume that $A \neq \alpha B$ for any real α. We are looking for

$$\min_{\alpha \in \mathbf{R}} \|A - \alpha B\| = d. \tag{1}$$

Here $\|A\|$ is the spectral norm of A, i.e., the square root of the greatest eigenvalue of A^*A. The special case where A is a real $n \times n$-matrix and B is the identity matrix I is interesting in the theory of partial differential equations [1]. It is well known that the set of all real β, such that

$$\|A - \beta B\| = d, \tag{2}$$

is a compact convex set. That is, this set is of the form

$$\alpha_1 \leq \beta \leq \alpha_2 \qquad (\alpha_1 \leq \alpha_2). \tag{3}$$

In what follows, we characterize α_1 and α_2 as solutions of certain polynomial equations.

2. The main result. Let

$$S(\lambda, \alpha) = \lambda^n + s_1(\alpha)\lambda^{n-1} + \cdots + s_n(\alpha) \tag{4}$$

be the characteristic polynomial of the matrix $(A^* - \alpha B^*)(A - \alpha B)$. Thus we consider $S(\lambda, \alpha)$ as a polynomial in λ over the polynomial ring $P[\alpha]$. Let $D_1(\lambda, \alpha)$ be the greatest common divisor of $S(\lambda, \alpha)$ and $\partial S(\lambda, \alpha)/\partial\lambda$ over the field $F[\alpha]$ of all rational functions in α. However, as the leading coefficient of $S(\lambda, \alpha)$ and $\partial S(\lambda, \alpha)/\partial\lambda$ do not depend on α, it is easy to show that $D_1(\lambda, \alpha)$ can be chosen as a polynomial over $P[\alpha]$, whose leading

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coefficient does not depend on α. By $R(\lambda, \alpha)$, we denote the reduced characteristic polynomial

$$R(\lambda, \alpha) = S(\lambda, \alpha)/D_1(\lambda, \alpha). \tag{5}$$

Again, $R(\lambda, \alpha)$ is a polynomial over $P[\alpha]$. Thus if $S(\lambda_0, \alpha_0) = 0$, then $R(\lambda_0, \alpha_0) = 0$. By the definition of $R(\lambda, \alpha)$, the discriminant $\rho(\alpha)$ of $R(\lambda, \alpha)$, i.e., the resultant of the polynomials $R(\lambda, \alpha)$ and $\partial R(\lambda, \alpha)/\partial\lambda$, is a nonzero element in $P[\alpha]$. Denote by $D_2(\lambda)$ the greatest common divisor of $R(\lambda, \alpha)$ and $R(\lambda, \alpha_0)$ over $F[\alpha]$. It means that the roots of $D_2(\lambda)$ are all the roots of $R(\lambda, \alpha)$ which do not depend on α. By $Q(\lambda, \alpha)$, we denote the polynomial

$$Q(\lambda, \alpha) = R(\lambda, \alpha)/D_2(\lambda) \tag{6}$$

over $P[\alpha]$.

Theorem. *Let A and B be $m \times n$-complex-valued nonzero matrices. Assume that $A \neq \alpha B$ for any real α. Then the real numbers α_1 and α_2, which are defined by the conditions* (2) *and* (3), *satisfy either the equation*

$$\rho(\alpha) = 0 \tag{7}$$

or the system

$$Q(\lambda, \alpha) = \partial Q(\lambda, \alpha)/\partial\alpha = 0. \tag{8}$$

The equation (7) *and the system* (8) *have only a finite number of solutions.*

3. **Proof.** Assume first that $\alpha_1 < \alpha_2$. So $\|A - \beta B\| = d$ for $\alpha_1 \leq \beta \leq \alpha_2$. This implies that $S(d^2, \alpha) = R(d^2, \alpha) = 0$ identically. We claim that, for $\alpha = \alpha_j$, $j = 1, 2$, d^2 is at least a double root of $R(\lambda, \alpha)$. Suppose that d^2 is a simple root for $\alpha = \alpha_j$. From (2), we deduce that all other roots of $R(\lambda, \alpha_j)$ are strictly less than d^2. Thus if $|\alpha - \alpha_j| < \epsilon$, d^2 is the greatest root of $R(\lambda, \alpha)$ for an appropriate positive ϵ. So $\|A - \alpha B\| = d$ for that α. This contradicts the definition of α_1 and α_2. Thus in the case that $\alpha_1 < \alpha_2$, α_j has to satisfy (7) for $j = 1, 2$. Since $\rho(\alpha)$ is a nontrivial polynomial, there exist only finite numbers of solutions. (It may happen that $\rho(\alpha) \equiv$ Const, which means that $\alpha_1 = \alpha_2$.) Assume now that $\alpha_1 = \alpha_2 = \alpha^*$. If d^2 is at least a double root of $R(\lambda, \alpha^*)$, then α^* has to satisfy (7). Suppose now that d^2 is a simple root of $R(\lambda, \alpha^*)$. Thus for $|\alpha - \alpha^*| < \epsilon$, there exists an analytic function $\lambda(\alpha)$ which is defined by the equation

$$R(\lambda, \alpha) = 0 \tag{9}$$

and $\lambda(\alpha^*) = d^2$. This is because $\partial R(\lambda, \alpha)/\partial\lambda \neq 0$ on this neighborhood of (d^2, α^*). We restrict ourselves to this neighborhood. Obviously, $\lambda(\alpha)$ is not a constant function. Otherwise, $\|A - \alpha B\| = d$, which contradicts the assumption that $\alpha_1 = \alpha_2 = \alpha^*$. So $\lambda(\alpha)$ is the greatest root of $Q(\lambda, \alpha) = 0$. Since $\lambda(\alpha) > d^2$ for $\alpha \neq \alpha^*$, we conclude that $d\lambda(\alpha^*)/d\alpha = 0$. This proves that α^* satisfies the system (8).

We show that (8) has only a finite number of solutions. First, note that all roots of $Q(\lambda, \alpha)$ are distinct except for a finite number of α. This follows from the fact that $\rho(\alpha)$ is a nonzero polynomial. Thus if the system (8) has an infinite number of solutions, it would imply that the polynomials $Q(\lambda, \alpha)$ and $\partial Q(\lambda, \alpha)/\partial\alpha$ have a nontrivial common factor $D_3(\lambda, \alpha)$. But this means that $Q(\lambda, \alpha)$ has constant roots which do not depend on α. This contradicts the definition of Q.

4. **Open problem.** Let A and $B_1, \cdots, B_k$ be $m \times n$-complex-valued matrices. Assume that $B_1, \cdots, B_k$ are linearly independent on $\mathbf{R}$. Denote by $\mathfrak{U}$ the real span of $B_1, \cdots, B_k$. An obvious extension of the problem discussed above is: Characterize the extreme points of the convex set of the best approximants $\mathfrak{B}$. That is, $\mathfrak{B} \subset \mathfrak{U}$, and, for any $B \in \mathfrak{U}$ and $C \in \mathfrak{B}$,

$$\|A - B\| \geq \|A - C\| = d. \tag{10}$$

It seems that, from the case $k = 1$, the extreme points of $\mathfrak{B}$ can be characterized by a set of certain polynomial equations.

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