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APPLICATIONS OF GRAPH THEORY TO MATRIX THEORY

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ABSTRACT. Let $A_1, \ldots, A_k$ be $n \times n$ matrices over a commutative ring R with identity. Graph theoretic methods are established to compute the standard polynomial $[A_1, \ldots, A_k]$. It is proved that if $k < 2n - 2$, and if the characteristic of R either is zero or does not divide $4I(\frac{1}{2}n) - 2$, where I denotes the greatest integer function, then there exist $n \times n$ skew-symmetric matrices $A_1, \ldots, A_k$ such that $[A_1, \ldots, A_k] \neq 0$.

1. Introduction. Let $A_1, \ldots, A_k$ be $n \times n$ matrices over a commutative ring R with identity. Let S_k be the symmetric group of degree k. Define the standard polynomial $[A_1, \ldots, A_k]$ by

$$[A_1, \ldots, A_k] = \sum \operatorname{sgn}(\sigma) A_{\sigma 1} \cdots A_{\sigma k},$$

where the summation is over all permutations $\sigma \in S_k$.

Amitsur and Levitzki proved algebraically [1] that $[A_1, \ldots, A_k] = 0$ if $k \geq 2n$. Their proof is elementary but lengthy. Swan gave a simpler and shorter graph theoretic proof of their theorem [8], [9]. Amitsur and Levitzki also proved that if $k < 2n$, then there exist $n \times n$ matrices $A_1, \ldots, A_k$ such that $[A_1, \ldots, A_k] \neq 0$, i.e., their theorem is sharp. See [1], [7] and [8] for examples. It is known [7] that if $k < 2n$, then there exist $n \times n$ symmetric matrices $A_1, \ldots, A_k$ such that $[A_1, \ldots, A_k] \neq 0$.

Kostant proved in [3] that $[A_1, \ldots, A_k] = 0$ if $k \geq 2n - 2$, where n is even, and each of the matrices A_j is complex skew-symmetric. In this paper we prove using graph theoretic methods that if $k < 2n - 2$, and if the characteristic of R either equals 0 or does not divide $4I(\frac{1}{2}n) - 2$, where I denotes the greatest integer function, then there exist $n \times n$ skew-symmetric matrices $A_1, \ldots, A_k$ such that $[A_1, \ldots, A_k] \neq 0$. This solves Conjecture 2 in [7] in the affirmative. In particular, for n even this implies that Kostant's theorem is sharp. Kostant's proof is nonelementary and uses cohomology theory. In [4] we present a graph theoretic proof that $[A_1, \ldots, A_k] = 0$ for $k \geq 2n - 2$

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if each of the matrices A_j is skew-symmetric, and R is an integral domain not of characteristic 2. This generalization of Kostant's theorem solves Conjecture 1 in [7] in the affirmative. The solutions of Conjectures 1 and 2 have been obtained independently by Hutchinson [2] and by Rowen [6]. Our results in this paper are somewhat stronger for the case when the characteristic of R is larger than 2. See [2], [5] and [6] for related results.

2. **Algebraic preliminaries.** We first state some algebraic properties of $[\ ,\dots,\ ]$. $[\ ,\dots,\ ]$ is alternating and multilinear. $[\ ,\dots,\ ]$ may be defined recursively by $[A] = A$ and

$$[A_1, \dots, A_k] = \sum_{j=1}^{k} (-1)^{j-1} A_j [A_1, \dots, \hat{A}_j, \dots, A_k]$$

$$= \sum_{j=1}^{k} (-1)^{k-j} [A_1, \dots, \hat{A}_j, \dots, A_k] A_j \quad \text{for } k > 1,$$

where the notation $\hat{A}_j$ means that the matrix A_j is absent.

The following propositions are easily established using the alternating and multilinear properties of $[\ ,\dots,\ ]$. Let A' denote the transpose of the matrix A.

Proposition 1. *If $A_1, \dots, A_k$ are $n \times n$ matrices, then*
(a) $[A_1, \dots, A_k]' = [A'_k, \dots, A'_1]$.
(b) $[A_1, \dots, A_k] = (-1)^{k(k-1)/2} [A_k, \dots, A_1]$.
(c) $[A_1, \dots, A_k]' = (-1)^{k(k-1)/2} [A'_1, \dots, A'_k]$.

Proposition 2. *If each A_j is an $n \times n$ skew-symmetric matrix, then*
(a) $[A_1, \dots, A_k]' = (-1)^{k(k+1)/2} [A_1, \dots, A_k]$.
(b) $[A_1, \dots, A_k]$ *is symmetric iff* $k \equiv 0$ *or* 3 (mod 4) *and is skew-symmetric iff* $k \equiv 1$ *or* 2 (mod 4).

Proposition 3. *If each A_j is an $n \times n$ symmetric matrix, then*
(a) $[A_1, \dots, A_k]' = (-1)^{k(k-1)/2} [A_1, \dots, A_k]$.
(b) $[A_1, \dots, A_k]$ *is symmetric iff* $k \equiv 0$ *or* 1 (mod 4) *and is skew-symmetric iff* $k \equiv 2$ *or* 3 (mod 4).

Let e_{ij} be the elementary matrix unit which has a 1 in the (i, j)th position and zeros elsewhere. Let s_{ij}, $1 \le i < j \le n$, denote the $n \times n$ skew-symmetric matrix unit $e_{ij} - e_{ji}$. Let t_{ij}, $1 \le i \le j \le n$, denote the $n \times n$ symmetric matrix unit $e_{ij} + (1 - \delta_{ij}) e_{ji}$, where δ_{ij} is the Kronecker delta. Then

$$e_{ij}e_{hl} = \delta_{jh}e_{il}, \tag{2.1}$$

$$\begin{aligned} s_{ij}s_{hl} = (e_{ij} - e_{ji})(e_{hl} - e_{lh}) &= e_{ij}e_{hl} - e_{ij}e_{lh} - e_{ji}e_{hl} + e_{ji}e_{lh} \\ &= \delta_{jh}e_{il} - \delta_{jl}e_{ih} - \delta_{ih}e_{jl} + \delta_{il}e_{jh}. \end{aligned}$$

$$= \begin{cases} 0 & \text{if } j < h, \\ e_{il} & \text{if } j = h, \\ -\delta_{ih}e_{jl} & \text{if } h < j < l, \\ -e_{ih} - \delta_{ih}e_{jj} & \text{if } j = l, \\ -\delta_{ih}e_{jl} + \delta_{il}e_{jh} & \text{if } j > l, \end{cases} \tag{2.2}$$

$$\begin{aligned} t_{ij}t_{hl} &= (e_{ij} + (1 - \delta_{ij})e_{ji})(e_{hl} + (1 - \delta_{hl})e_{lh}) \\ &= e_{ij}e_{hl} + (1 - \delta_{hl})e_{ij}e_{lh} + (1 - \delta_{ij})e_{ji}e_{hl} + (1 - \delta_{ij})(1 - \delta_{hl})e_{ji}e_{lh} \\ &= \delta_{jh}e_{il} + (1 - \delta_{hl})\delta_{jl}e_{ih} + (1 - \delta_{ij})\delta_{ih}e_{jl} + (1 - \delta_{ij})(1 - \delta_{hl})\delta_{il}e_{jh} \end{aligned}$$

$$= \begin{cases} 0 & \text{if } j < h, \\ e_{il} & \text{if } j = h, \\ \delta_{ih}e_{jl} & \text{if } h < j < l, \\ \delta_{jh}e_{ij} + (1 - \delta_{hj})e_{ih} + (1 - \delta_{ij})\delta_{ih}e_{jj} & \text{if } j = l, \\ \delta_{ih}e_{jl} + (1 - \delta_{hl})\delta_{il}e_{jh} & \text{if } j > l. \end{cases} \tag{2.3}$$

3. **Graph theoretic preliminaries.** Let G be a graph having n vertices $v_1, \ldots, v_n$ and k edges $e_1, \ldots, e_k$. If v_i and v_j are vertices of G, then an Euler path in G from v_i to v_j is a permutation $\omega \in S_k$ for which there exists an orientation of G such that

(a) $e_{\omega 1}$ starts at v_i, i.e., v_i is the initial vertex of $e_{\omega 1}$,

(b) $e_{\omega k}$ ends at v_j, i.e., v_j is the terminal vertex of $e_{\omega k}$, and

(c) the terminal vertex of $e_{\omega h}$ is the initial vertex of $e_{\omega(h+1)}$ for $1 \leq h < k$.

If, in addition, G is a digraph, then a unicursal path [8] in G from v_i to v_j is an Euler path in G from v_i to v_j with respect to the given orientation of G. Thus if G is a digraph, every unicursal path in G from v_i to v_j is also an Euler path in G from v_i to v_j but not conversely, i.e. we deal only with the given directions of the edges when considering unicursal paths.

If G is a digraph without loops and ω is an Euler path in G from v_i to v_j, then some of the edges of G may have directions induced by ω opposite to their given directions. We refer to the number of such edges by $r(\omega)$. Thus the number of edges of G which have directions induced by ω which are the same as their given directions is $k - r(\omega)$.

Let each A_j be some e_{hl}. Define a digraph G as follows. G has n vertices $v_1, \dots, v_n$, and G has a directed edge e_j from v_h to v_l for each $A_j = e_{hl}$. The following theorem is due to Swan [8] and is immediate from multiplication rule (2.1).

Theorem 1. *The (i, j)th entry in $[A_1, \dots, A_k]$ is $\Sigma\, \mathrm{sgn}(\omega)$, where the summation is over all unicursal paths ω in G from v_i to v_j.*

Let each A_j be some s_{hl}. Define a digraph G as follows. G has n vertices $v_1, \dots, v_n$, and G has a directed edge e_j from v_h to v_l for each $A_j = s_{hl}$. The next theorem follows immediately from Theorem 1 and multiplication rule (2.2).

Theorem 2. *The (i, j)th entry in $[A_1, \dots, A_k]$ is $\Sigma(-1)^{r(\omega)}\, \mathrm{sgn}(\omega)$, where the summation is over all Euler paths ω in G from v_i to v_j.*

Let each A_j be some t_{hl}. Define a graph G as follows. G has n vertices $v_1, \dots, v_n$, and G has an edge e_j from v_h to v_l for each $A_j = t_{hl}$. The next theorem follows immediately from Theorem 1 and multiplication rule (2.3).

Theorem 3. *The (i, j)th entry in $[A_1, \dots, A_k]$ is $\Sigma\, \mathrm{sgn}(\omega)$, where the summation is over all Euler paths ω in G from v_i to v_j.*

It is easy to give a similar graph theoretic interpretation to $[A_1, \dots, A_k]$ when the A_j's are a mixture of elementary, skew-symmetric and symmetric matrix units.

4. Main result. This section establishes

Theorem 4. *If $k < 2n - 2$, and if the characteristic of R either equals 0 or does not divide $4I(\frac{1}{2}n) - 2$, where I denotes the greatest integer function, then there exist $n \times n$ skew-symmetric matrices $A_1, \dots, A_k$ such that $[A_1, \dots, A_k] \neq 0$.*

By the recursion formula for $[A_1, \dots, A_k]$ in §2 it is sufficient to prove Theorem 4 for the case $k = 2n - 3$, $n > 1$. For this case let the matrices $A_1, \dots, A_k$ be $s_{12}, s_{23}, s_{13}, s_{34}, s_{24}, \dots, s_{n-1,n}, s_{n-2,n}$, and let $B_n = [A_1, \dots, A_k]$. By direct computation $B_2 = s_{12}$ and $B_3 = -2(e_{11} + e_{22} + e_{33})$. Hence, Theorem 4 is true for $n = 2$ or 3. Figure 1 illustrates a portion of the digraph G_n described preceding Theorem 2 associated with B_n.

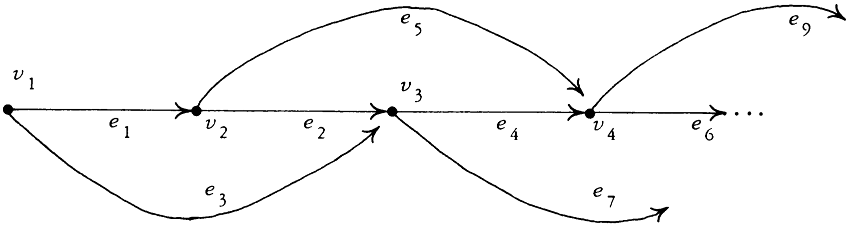

Figure 1. G_n, $n > 1$, has n vertices and $2n - 3$ edges.

Lemma 1. *If $n > 3$, then there exists an integer c_n such that $B_n = c_n(e_{2,n-1} + (-1)^{n-1}e_{n-1,2})$, i.e., $B_n = c_n s_{2,n-1}$ for n even > 3, and $B_n = c_n t_{2,n-1}$ for n odd > 3.*

Proof. Proposition 2 implies that B_n is skew-symmetric iff n is even and is symmetric iff n is odd. v_2 and v_{n-1} are the only vertices of G_n of odd order since $n > 3$. Hence, the only possible Euler paths ω in G_n are from v_2 to v_{n-1} or from v_{n-1} to v_2. Apply Theorem 2.

The first few computed c_n's are $c_4 = c_5 = -6$, $c_6 = c_7 = -10$ and $c_8 = -14$. For later convenience we set $c_2 = c_3 = -2$.

This paragraph is not necessary for the proof of the theorem but may be of interest. If E_n denotes the number of Euler paths ω in G_n from v_2 to v_{n-1}, then $E_2 = 1$, $E_3 = 2$, $E_4 = 6$, $E_5 = 16$, $E_6 = 44$, $E_7 = 120$ and $E_8 = 328$. We set $E_1 = 0$. Then the number of such Euler paths

(a) with $\omega 1 = 1$ is E_{n-1} for $n > 1$,

(b) with $\omega 1 = 2$ is E_{n-1} for $n > 1$, and

(c) with $\omega 1 = 5$ is $2E_{n-2}$ for $n > 2$.

(a), (b) and (c) imply that the E_n's satisfy the difference equation $E_n = 2E_{n-1} + 2E_{n-2}$ for $n > 2$ with the initial conditions $E_1 = 0$ and $E_2 = 1$ whose solution is

$$E_n = (\sqrt{3}/6)((1 + \sqrt{3})^{n-1} - (1 - \sqrt{3})^{n-1}).$$

Lemma 2. *Let P_n denote the set of all Euler paths in G_n from v_2 to v_{n-1}. Then $\Sigma(-1)^{r(\omega)}\operatorname{sgn}(\omega)$ equals*

(a) *c_{n-1} for $n > 3$, where the summation is over all $\omega \in P_n$ such that $\omega 1 = 1$,*

(b) *$c_{n-2} - c_{n-3}$ for $n > 4$, where the summation is over all $\omega \in P_n$ such that $\omega 1 = 2$,*

(c) *0 for $n > 4$, where the summation is over all $\omega \in P_n$ such that $\omega 1 = 5$.*

Proof of (a). $\omega 2 = 3$ for each $\omega \in P_n$ such that $\omega 1 = 1$. Define f_j by $f_j(x) = x + j$ and $f: \{2, 4, 5, \ldots, 2n-3\} \to \{1, 2, 3, \ldots, 2n-5\}$ by $f(2)=1$ and $f(x) = x - 2$ for $4 \leq x \leq 2n-3$. Define $F: \{\omega \in P_n | \omega 1 = 1\} \to P_{n-1}$ by $\omega' = F(\omega) = f \circ \omega \circ f_2$. F is a 1-1 correspondence such that $r(\omega) = r(\omega') + 1$ and $\mathrm{sgn}(\omega) = -\mathrm{sgn}(\omega')$ for each $\omega \in P_n$ such that $\omega 1 = 1$. Therefore, $c_{n-1} = \Sigma(-1)^{r(\omega')} \mathrm{sgn}(\omega')$, where the summation is over all $\omega' \in P_{n-1}$, $= \Sigma(-1)^{r(\omega)} \mathrm{sgn}(\omega)$, where the summation is over all $\omega \in P_n$ such that $\omega 1 = 1$.

Proof of (b). By direct computation $\Sigma(-1)^{r(\omega)} \mathrm{sgn}(\omega) = 0 = c_3 - c_2$, where the summation is over all $\omega \in P_5$ such that $\omega 1 = 2$. Hence, we may assume that $n > 5$. $\omega 2 = 3$, 4 or 7 for each $\omega \in P_n$ such that $\omega 1 = 2$. $\omega 3 = 1$ and $\omega 4 = 5$ for each $\omega \in P_n$ such that $\omega 1 = 2$ and $\omega 2 = 3$. Define $f: \{4, 6, 7, \ldots, 2n-3\} \to \{1, 2, 3, \ldots, 2n-7\}$ by $f(4) = 1$ and $f(x) = x - 4$ for $6 \leq x \leq 2n-3$. Define $F: \{\omega \in P_n | \omega 1 = 2 \text{ and } \omega 2 = 3\} \to P_{n-2}$ by $\omega' = F(\omega) = f \circ \omega \circ f_4$. F is a 1-1 correspondence such that $r(\omega) = r(\omega') + 1$ and $\mathrm{sgn}(\omega) = -\mathrm{sgn}(\omega')$ for each $\omega \in P_n$ such that $\omega 1 = 2$ and $\omega 2 = 3$. Therefore, $c_{n-2} = \Sigma(-1)^{r(\omega')} \mathrm{sgn}(\omega')$, where the summation is over all $\omega' \in P_{n-2}$, $= \Sigma(-1)^{r(\omega)} \mathrm{sgn}(\omega)$, where the summation is over all $\omega \in P_n$ such that $\omega 1 = 2$ and $\omega 2 = 3$.

$\omega 3 = 5$, 6 or 9 for each $\omega \in P_n$ such that $\omega 1 = 2$ and $\omega 2 = 4$. $\omega 4 = 1$, $\omega 5 = 3$ and $\omega 6 = 7$ for each $\omega \in P_n$ such that $\omega 1 = 2$, $\omega 2 = 4$ and $\omega 3 = 5$. Define $f: \{6, 8, 9, \ldots, 2n-3\} \to \{1, 2, 3, \ldots, 2n-9\}$ by $f(6) = 1$ and $f(x) = x - 6$ for $8 \leq x \leq 2n-3$. Define $F: \{\omega \in P_n | \omega 1 = 2, \omega 2 = 4 \text{ and } \omega 3 = 5\} \to P_{n-3}$ by $\omega' = F(\omega) = f \circ \omega \circ f_6$. F is a 1-1 correspondence such that $r(\omega) = r(\omega') + 2$ and $\mathrm{sgn}(\omega) = \mathrm{sgn}(\omega')$ for each $\omega \in P_n$ such that $\omega 1 = 2$, $\omega 2 = 4$, and $\omega 3 = 5$. Therefore, $c_{n-3} = \Sigma(-1)^{r(\omega')} \mathrm{sgn}(\omega')$, where the summation is over all $\omega' \in P_{n-3}$, $= \Sigma(-1)^{r(\omega)} \mathrm{sgn}(\omega)$, where the summation is over all $\omega \in P_n$ such that $\omega 1 = 2$, $\omega 2 = 4$ and $\omega 3 = 5$.

If $\omega 1 = 2$, $\omega 2 = 4$ and $\omega 3 = 6$, then v_5 and v_6 are connected by two subpaths of ω, namely e_8 and the subpath consisting of the edges e_7, e_3, e_1, e_5 and e_9. Either exactly two or exactly three of these six edges have orientations induced by ω opposite to their given orientations. We may place the set of all $\omega \in P_n$ such that $\omega 1 = 2$, $\omega 2 = 4$ and $\omega 3 = 6$ into 1-1 correspondence with itself by interchanging the order of these two subpaths if exactly 2 of the 6 edges have orientations induced by ω opposite to their given orientations, and by interchanging the order of these two subpaths and reversing their induced orientations if exactly 3 of the 6 edges have orientations induced by ω opposite to their given orientations. If $\omega \leftrightarrow \omega'$ denotes this correspondence, then $r(\omega) = r(\omega')$ and $\mathrm{sgn}(\omega) = -\mathrm{sgn}(\omega')$. Therefore, $\Sigma(-1)^{r(\omega)} \mathrm{sgn}(\omega) = -\Sigma(-1)^{r(\omega)} \mathrm{sgn}(\omega)$, and so $\Sigma(-1)^{r(\omega)} \mathrm{sgn}(\omega) = 0$, where each summation is over all $\omega \in P_n$ such that $\omega 1 = 2$, $\omega 2 = 4$ and $\omega 3 = 6$.

We may place the set of all $\omega \in P_n$ such that $\omega 1 = 2$, $\omega 2 = 4$ and $\omega 3 = 9$ into 1-1 correspondence with itself by reversing the induced orientation of the cycle consisting of the edges e_7, e_3, e_1, e_5 and e_6 for each $\omega \in P_n$ such that $\omega 1 = 2$, $\omega 2 = 4$ and $\omega 3 = 9$. If $\omega \leftrightarrow \omega'$ denotes this correspondence, then $r(\omega) \equiv r(\omega') + 1 \pmod 2$ and $\operatorname{sgn}(\omega) = \operatorname{sgn}(\omega')$. Therefore, $\Sigma(-1)^{r(\omega)} \operatorname{sgn}(\omega) = -\Sigma(-1)^{r(\omega)} \operatorname{sgn}(\omega)$, and so $\Sigma(-1)^{r(\omega)} \operatorname{sgn}(\omega) = 0$, where each summation is over all $\omega \in P_n$ such that $\omega 1 = 2$, $\omega 2 = 4$ and $\omega 3 = 9$.

If $\omega 1 = 2$ and $\omega 2 = 7$, then there exists i, $4 \le i \le 2n - 7$, such that $\omega(i + j) \in \{1, 3, 4, 5\}$ for $0 \le j \le 3$. Define $f: \{6, 8, 9, \ldots, 2n - 3\} \to \{1, 2, 3, \ldots, 2n - 9\}$ by $f(6) = 1$ and $f(x) = x - 6$ for $8 \le x \le 2n - 3$. Define $\omega' = F(\omega)$ by $\omega' h = f \circ \omega(h + 2)$ for $1 \le h < i - 2$ and $\omega' h = f \circ \omega(h + 6)$ for $i - 2 \le h \le 2n - 9$. $F: \{\omega \in P_n | \omega 1 = 2 \text{ and } \omega 2 = 7\} \to P_{n-3}$ is a 2-1 map such that $r(\omega) = r(\omega') + 2$ and $\operatorname{sgn}(\omega) = -\operatorname{sgn}(\omega')$ for each $\omega \in P_n$ such that $\omega 1 = 2$ and $\omega 2 = 7$. Therefore, $-2c_{n-3} = -2\Sigma(-1)^{r(\omega')} \operatorname{sgn}(\omega')$, where the summation is over all $\omega' \in P_{n-3}$, $= \Sigma(-1)^{r(\omega)} \operatorname{sgn}(\omega)$, where the summation is over all $\omega \in P_n$ such that $\omega 1 = 2$ and $\omega 2 = 7$.

Proof of (c). By direct computation $\Sigma(-1)^{r(\omega)} \operatorname{sgn}(\omega) = 0$, where the summation is over all $\omega \in P_5$ such that $\omega 1 = 5$. Hence, we may assume that $n > 5$. $\omega 2 = 4$, 6 or 9 for each $\omega \in P_n$ such that $\omega 1 = 5$. $\omega 6 = 7$ for each $\omega \in P_n$ such that $\omega 1 = 5$ and $\omega 2 = 4$. Define $f: \{6, 8, 9, \ldots, 2n - 3\} \to \{1, 2, 3, \ldots, 2n - 9\}$ by $f(6) = 1$ and $f(x) = x - 6$ for $8 \le x \le 2n - 3$. Define $F: \{\omega \in P_n | \omega 1 = 5 \text{ and } \omega 2 = 4\} \to P_{n-3}$ by $\omega' = F(\omega) = f \circ \omega \circ f_6$. F is a 2-1 map such that $r(\omega) = r(\omega') + 2$ and $\operatorname{sgn}(\omega) = \operatorname{sgn}(\omega')$ if $\omega 3 = 3$, $\omega 4 = 1$ and $\omega 5 = 2$, and $r(\omega) = r(\omega') + 3$ and $\operatorname{sgn}(\omega) = -\operatorname{sgn}(\omega')$ if $\omega 3 = 2$, $\omega 4 = 1$ and $\omega 5 = 3$ for each $\omega \in P_n$ such that $\omega 1 = 5$ and $\omega 2 = 4$. Therefore, $2c_{n-3} = 2\Sigma(-1)^{r(\omega')} \operatorname{sgn}(\omega')$, where the summation is over all $\omega' \in P_{n-3}$, $= \Sigma(-1)^{r(\omega)} \operatorname{sgn}(\omega)$, where the summation is over all $\omega \in P_n$ such that $\omega 1 = 5$ and $\omega 2 = 4$.

If $\omega 1 = 5$ and $\omega 2 = 6$, then there exists i, $4 \le i \le 2n - 7$, such that $\omega(i + j) \in \{1, 2, 3, 4\}$ for $0 \le j \le 3$. Define $f: \{7, 8, 9, \ldots, 2n - 3\} \to \{1, 2, 3, \ldots, 2n - 9\}$ by $f(x) = x - 6$. Define $\omega' = F(\omega)$ by $\omega' h = f \circ \omega(h + 2)$ for $1 \le h < i - 2$ and $\omega' h = f \circ \omega(h + 6)$ for $i - 2 \le h \le 2n - 9$. $F: \{\omega \in P_n | \omega 1 = 5 \text{ and } \omega 2 = 6\} \to P_{n-3}$ is a 2-1 map. If $\omega i = 2$, then $r(\omega) = r(\omega') + 2$ and $\operatorname{sgn}(\omega) = -\operatorname{sgn}(\omega')$. If $\omega i = 3$, then $r(\omega) = r(\omega') + 1$ and $\operatorname{sgn}(\omega) = \operatorname{sgn}(\omega')$. If $\omega i = 4$ and $\omega(i + 1) = 2$, then $r(\omega) = r(\omega') + 3$ and $\operatorname{sgn}(\omega) = \operatorname{sgn}(\omega')$. If $\omega i = 4$ and $\omega(i + 1) = 3$, then $r(\omega) = r(\omega') + 2$ and $\operatorname{sgn}(\omega) = -\operatorname{sgn}(\omega')$.

Thus, $(-1)^{r(\omega)} \operatorname{sgn}(\omega) = -(-1)^{r(\omega')} \operatorname{sgn}(\omega')$ for each $\omega \in P_n$ such that $\omega 1 = 5$ and $\omega 2 = 6$. Therefore, $-2c_{n-3} = -2\Sigma(-1)^{r(\omega')} \operatorname{sgn}(\omega')$, where the summation is over all $\omega' \in P_{n-3}$, $= \Sigma(-1)^{r(\omega)} \operatorname{sgn}(\omega)$, where the summation is over all $\omega \in P_n$ such that $\omega 1 = 5$ and $\omega 2 = 6$.

We may place the set of all $\omega \in P_n$ such that $\omega 1 = 5$ and $\omega 2 = 9$ into 1-1 correspondence with itself by reversing the induced orientation of the subpath consisting of the edges e_7, e_3, e_1, e_2, e_4 and e_6 for each $\omega \in P_n$ such that $\omega 1 = 5$ and $\omega 2 = 9$. If $\omega \leftrightarrow \omega'$ denotes this correspondence, then $r(\omega) \equiv r(\omega') \pmod 2$ and $\operatorname{sgn}(\omega) = -\operatorname{sgn}(\omega')$. Therefore, $\Sigma(-1)^{r(\omega)} \operatorname{sgn}(\omega) = -\Sigma(-1)^{r(\omega)} \operatorname{sgn}(\omega)$, and so $\Sigma(-1)^{r(\omega)} \operatorname{sgn}(\omega) = 0$, where each summation is over all $\omega \in P_n$ such that $\omega 1 = 5$ and $\omega 2 = 9$.

The next lemma is immediate from Lemma 2 and Theorem 2.

Lemma 3. *The c_n's satisfy the difference equation $c_n = c_{n-1} + c_{n-2} - c_{n-3}$ for $n > 4$ with the initial conditions $c_2 = c_3 = -2$ and $c_4 = -6$.*

We obtain by induction from Lemma 3 and $c_{n-1} \leq c_{n-2} \leq c_{n-3}$ for $n > 4$ that $c_n = c_{n-1} + (c_{n-2} - c_{n-3}) \leq c_{n-1}$. Thus $c_n \leq c_{n-1} < 0$ for $n > 2$. In fact, solving the difference equation with the initial conditions in Lemma 3 we obtain $c_n = 2 - 4I(\frac{1}{2}n)$ for $n > 1$. This completes the proof of Theorem 4. As a final remark the computations above are all valid if the A_j's are regarded as $m \times m$ matrices for any $m \leq n$.

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