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CLOSED MAPPINGS OF σ-LOCALLY COMPACT METRIC SPACES

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ABSTRACT. We show that a metric space M is σ-locally compact if and only if every image of M under a closed, continuous function is the countable union of closed, metrizable, locally compact subspaces.

Several other theorems about closed, continuous images of metric spaces are given; one of these is that the closed, continuous image of a complete, σ-locally compact metric space must contain a dense, metrizable open set.

1. **Introduction.** Interest has been shown recently in the circumstances under which the image of a metric space under a closed, continuous function is the countable union of closed, metrizable subspaces. This question apparently originated with Nagata [1, Question 4, p. 67]. Results have been given by Fitzpatrick [2] and Van Doren [3]. The purpose of this note is to give necessary and sufficient conditions that such a space be one such union.

Since the study of images of metric spaces under closed mappings is equivalent to the study of decomposition spaces of metric spaces generated by upper semicontinuous (usc) decompositions, we will use the terminology of usc decompositions except in certain statements of theorems.

If H is an usc decomposition of the metric space M, then $\hat{H}$ will denote the decomposition space generated by H. If $k \in H$, then $\hat{k}$ will denote the corresponding point of $\hat{H}$. $\hat{W}$ will denote the collection of all points of $\hat{H}$ at which $\hat{H}$ is not first countable. It is inherent in the proof of the frequently cited theorem of Stone [4] and Morita and Hanai [5] that the corresponding subset W of H is the collection of all elements of H having noncompact boundaries.

2. **Lemma.** *Let H be an usc decomposition of the metric space M. If $\hat{W}$ is a G_δ set, then $\hat{H}$ is the countable union of closed, metrizable subspaces.*

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Proof. Let $\hat{W}$ be the intersection of the sequence $\hat{D}_1, \hat{D}_2, \ldots$ of open sets of $\hat{H}$. Then for each positive integer i, the closed subset $\hat{H} - \hat{D}_i$ of $\hat{H}$ is first countable at each of its points, and so, by Theorem 1 of [4], metrizable. But Lašnev [6] has shown that $\hat{W}$ is contained in the union of a countable collection of closed discrete subspaces of $\hat{H}$ (by showing the existence of a σ-discrete subcollection of H containing W). Thus $\hat{H}$ is the countable union of closed, metrizable subspaces.

3. **Theorem.** *If H is an usc decomposition of the locally compact metric space M, then the collection W of elements of H having noncompact boundaries is discrete.*

Proof. This is a special case of a theorem of K. Morita (see the *proof* of Theorem 4 (a) of [7, p. 540]) who shows this for M locally compact, paracompact, and Hausdorff. Morita's theorem applies since Stone [8] has shown that every metric space is paracompact.

4. **Theorem.** *In order that the metric space M be σ-locally compact, it is necessary and sufficient that every closed, continuous image of M be the countable union of closed, metrizable, locally compact subspaces.*

Proof. Sufficiency is obtained trivially, since the identity map is closed and continuous.

To obtain necessity, we let H be as before. Let $M = C_1 + C_2 + \cdots$ where each C_i is locally compact. We first show that M is the countable union of *closed* locally compact subspaces by showing that each C_i is contained in such a union.

Given some C_i, let A be the collection of points of the subspace $\mathrm{cl}(C_i)$ of M at which $\mathrm{cl}(C_i)$ is not locally compact. If $p \in A$, then each open set of $\mathrm{cl}(C_i)$ containing p also contains a sequence of points which has no limit point in M. But then if $p \in C_i$, we must have (since C_i is dense in $\mathrm{cl}(C_i)$) that each open set of C_i containing p must also contain a sequence of points having no limit point in M and hence no limit point in C_i. This contradicts the local compactness of C_i, so $p \notin C_i$. Thus A does not intersect C_i. A is also closed in $\mathrm{cl}(C_i)$ and thus is closed in M. Hence the sequence $\{p \in \mathrm{cl}(C_i) \mid d(p, A) \geq 1/n\}_{n=1}^{\infty}$ where d is the distance function on M, is a sequence of closed, locally compact subspaces of M, the union of which covers C_i. We may therefore write $M = F_1 + F_2 + \cdots$ where each F_i is a closed, locally compact subspace of M.

We pause to observe that this proof depends on the fact that the closure of C_i in M is (as a subspace) locally compact at each point of C_i. Any condition on C_i which implies this also implies that M is the countable union of closed, locally compact subspaces, and thus the necessity in Theorem 4. One such condition on C_i which is weaker than local compactness is this: if $p \in C_i$, there is an open set R of M containing p such that the closure in M of $R \cap C_i$ is compact.

Returning to the proof, for each positive integer i, the set $H_i = \{k \cap F_i | k \in H\}$ is an usc decomposition of the locally compact metric space F_i. It follows from Lemma 2 and Theorem 3 that the decomposition space $\hat{H}_i$ of F_i generated by H_i is the countable union of closed (in $\hat{H}_i$), metrizable subspaces. We note that each $\hat{H}_i$ is a closed subspace of $\hat{H}$, and $\hat{H} = \hat{H}_1 + \hat{H}_2 + \cdots$, so it follows that $\hat{H}$ is the countable union of closed metrizable subspaces.

5. **Corollary.** *If the metric space M contains a σ-locally compact subspace F which is closed and intersects each element of the usc decomposition H of M, then the decomposition space $\hat{H}$ generated by H is the countable union of closed, metrizable subspaces.*

Proof. The set $V = \{k \cap F | k \in H\}$ is an usc decomposition of the σ-locally compact metric space F, and the decomposition space $\hat{V}$ is homeomorphic to $\hat{H}$.

6. **Corollary.** *The closed, continuous image of a complete, σ-locally compact metric space contains a dense, metrizable open set.*

Proof. Let H and M be as before. For each open set $\hat{D}$ of $\hat{H}$, let $H(\hat{D}) = \{k \in H | \hat{k} \in \hat{D}\}$. Let $D = \bigcup H(\hat{D})$. D is an open, σ-locally compact, topologically complete subspace of M, and $H(\hat{D})$ is an usc decomposition of D. Van Doren [3] shows that if, in the closed, continuous image T of a complete metric space, the collection of points of T at which T is not first countable is dense, then T is not the countable union of closed, metrizable subspaces. The decomposition space generated by $H(\hat{D})$ (which space is identical with $\hat{D}$) is, by Theorem 4, such a union, so $\hat{D}$ must contain an open subspace at each point of which $\hat{D}$ is first countable. The union of all such open subspaces, over all open subsets of $\hat{H}$ is a dense, open, metrizable subspace of $\hat{H}$.

7. **Remarks.** I do not know whether or not the sufficiency in Theorem 4 holds if the condition of local compactness is removed from the elements of the countable union.

There are examples of usc decomposition spaces of complete metric spaces ([2] implicitly) and, respectively, σ-compact metric spaces which have dense, nowhere first countable subspaces, so the conclusions of Theorem 3 and Corollary 6 cannot be extended.

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