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A CATEGORY THEOREM FOR TSUJI FUNCTIONS

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ABSTRACT. If H denotes the functions analytic in the open unit disk with the topology of uniform convergence on compact subsets, both the Tsuji functions in H and the functions in H with nonempty Tsuji sets comprise sets of first category in H. A question is posed about the category of a class of functions containing the Tsuji functions.

1. Introduction. Let $D = \{|z| < 1\}$, $C = \{|z| < 1\}$, and H be the collection of functions analytic in D with the topology of uniform convergence on compact subsets of D. For each $f \in H$ and $z \in D$ let $f^*(z) = |f'(z)|/(1 + |f(z)|^2)$, the spherical derivative of f at z. For each r, $0 < r < 1$, and each $f \in H$, we let $L(f, r) = \int_0^{2\pi} r f^*(re^{i\phi})\,d\phi$. If $\limsup_{r \to 1^-} L(f, r) < \infty$, f is called a Tsuji function. (First introduced in [4], the Tsuji functions have since been extensively studied [1], [2], [3].)

If, for each $\alpha \in D$, we let $\phi_\alpha(z) = (z - \alpha)/(1 - \bar{\alpha}z)$, the Tsuji set of $f \in H$ is the set of points $\alpha \in H$ for which $f \circ \phi_\alpha$ is a Tsuji function. Tsuji sets were defined in [2], and they have not yet been characterized. In this note we prove the following

Theorem. *The collection of functions in H which have a nonempty Tsuji set is of first category in H.*

This result also shows that the Tsuji functions in H are of first category in H, which strengthens a result proved by F. Bagemihl [1].

2. Proof of the Theorem. Letting $\mathcal{T}$ be the collection of functions in H having a nonempty Tsuji set, we will show that $\mathcal{T}$ is a countable union of sets which are closed and nowhere dense in H. If $f \in \mathcal{T}$, then for some $\alpha \in D$, $x > 0$, and $y \in (0, 1)$, $L(f \circ \phi_\alpha, r) \leq x$ for all $r \in (y, 1)$. For each triple (n, m, k) of positive integers let $T(n, m, k)$ be the set of functions in H for which there exists $\alpha \in D$, $|\alpha| \leq 1 - 1/n$, such that $L(f \circ \phi_\alpha, r) \leq m$ for all $r \in (1 - 1/(k + 1), 1)$. It is clear that $\mathcal{T} = \bigcup_{(n,m,k)} T(n, m, k)$, the

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union being taken over the triples described.

To prove that each $T(n, m, k)$ is closed in H, we first state a preparatory lemma.

Lemma 1. *Let* $\{\alpha_j\}_{j=1}^{\infty}$ *be a sequence in* D *with* $\alpha_j \to \alpha \in D$. *Let* $\phi(z) = (z - \alpha)/(1 - \bar{\alpha}z)$, *and for each* j, $\phi_j(z) = (z - \alpha_j)/(1 - \bar{\alpha}_j z)$. *For a sequence* $\{f_j\}_{j=1}^{\infty} \subset H$ *with* $f_j \to f$ *in* H:

(i) $\phi_j \to \phi$ *in* H,

(ii) $f_j \circ \phi_j \to f \circ \phi$ *in* H,

(iii) $\{(f_j \circ \phi_j)^*\}_1^{\infty}$ *converges to* $(f \circ \phi)^*$ *uniformly on compact subsets of* D,

(iv) *for each* $r \in (0, 1)$, $L(f_j \circ \phi_j, r) \to L(f \circ \phi, r)$.

Lemma 2. *Each* $T(n, m, k)$ *is closed in* H.

Proof. Let $\{f_j\}$ be a sequence in $T(n, m, k)$ with $f_j \to f$ in H. For each j there is a point $\alpha_j \in D$, $|\alpha_j| \leq 1 - 1/n$, such that $L(f_j \circ \phi_j, r) \leq m$ when $r \in (1 - 1/(k + 1), 1)$. We may suppose $\alpha_j \to \alpha$, where $|\alpha| \leq 1 - 1/n$, and let $\phi(z) = (z - \alpha)/(1 - \bar{\alpha}z)$. Lemma 1(iv) shows that $L(f \circ \phi, r) \leq m$ for each $r \in (1 - 1/(k + 1), 1)$, so that $f \in T(n, m, k)$.

Lemma 3. *Each* $T(n, m, k)$ *is nowhere dense in* H.

Proof. For an arbitrary $f \in T(n, m, k)$ we shall show there exists a sequence in $H - T(n, m, k)$ which converges in H to f. Since $T(n, m, k)$ is closed, this will show it is nowhere dense in H.

For some $\alpha \in D$, $|\alpha| \leq 1 - 1/n$, $L(f \circ \phi_\alpha, r) \leq m$ for all $r \in (1 - 1/(k + 1), 1)$. For each positive integer q let S_q be the qth partial sum of the Maclaurin's series for f.

Given q, let $p(q)$ be a positive integer, and define $g_q(z) = S_q(z) + z^{p(q)}$. As long as $\{p(q)\}_{q=1}^{\infty}$ is increasing, $g_q \to f$ in H. If $p(q)$ is sufficiently large, on C both $|g_q'| > (q + p(q))/2$ and $|g_q| \leq |S_q| + 1$. Thus we may take $\rho(q) \in (0, 1)$ so that every Jordan curve in the annulus $\rho(q) < |z| < 1$ whose interior contains 0 is mapped by g_q onto a closed curve of spherical length at least $p(q)$. If $p(q)$ is sufficiently large and $\rho(q)$ is near enough to 1, we will have $L(g_q \circ \phi_\alpha, r) > m$ for a value of $r > 1 - 1/(k + 1)$.

With suitable choice of the sequence $\{p(q)\}_{q=1}^{\infty}$, the sequence $\{g_q\}_{q=1}^{\infty}$ lies in $H - T(n, m, k)$ and converges to f in H.

3. In his paper on Tsuji functions [3], W. K. Hayman introduces a larger related class of functions. A function $f \in H$ lies in class T_2 if

there exists a sequence $\{J_n\}_1^\infty$ of Jordan curves in D such that: (i) $J_n \subset \operatorname{int} J_{n+1}$; (ii) $\min_{J_n} |z| \to 1$ as $n \to \infty$; (iii) $\limsup_{n\to\infty} \int_{J_n} f^*(z)\,|dz| < \infty$.

The class T_2 contains the functions in H with nonempty Tsuji set, so the following question is natural.

Question. Is the class T_2 of first category in H?

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