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A MAXIMAL REALCOMPACTIFICATION WITH 0-DIMENSIONAL OUTGROWTH

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ABSTRACT. If X is a rimhard space, then it has a realcompactification with properties similar to those of the Freudenthal compactification of a rimcompact space.

1. **Rimhard spaces.** All spaces discussed in this paper are assumed completely regular and Hausdorff, and all compactifications of such spaces are assumed Hausdorff. All spaces are to be nonrealcompact unless explicitly stated otherwise.

Several times below, we shall be concerned with a family of extensions of a space X. In each case we shall identify two members of the family if there is a homeomorphism from one onto the other which fixes X pointwise, and we shall partially order the resulting collection of equivalence classes by $\alpha X \leq \gamma X$ if there is a continuous map from γX onto αX which fixes X pointwise.

The particular case where the family of extensions is the upper semilattice $\mathcal{K}(X)$ of compactifications of a space X has been well studied [1], [8], [9], [12]. $\mathcal{K}(X)$ has the Stone-Čech compactification βX as its largest element and it has a smallest element, the Alexandrov compactification X^*, if and only if X is locally compact. In the latter case, the lattice structure of $\mathcal{K}(X)$ has been shown in [7] to characterize the topological structure of $\beta X - X$.

If X is a rimcompact space, i.e. has a basis of open sets with compact frontiers, then $\mathcal{K}(X)$ contains the Freudenthal compactification FX. This is characterized [4], [10] by the properties that $FX - X$ is 0-dimensional (has a basis of open-closed sets) and if αX is any compactification such that $\alpha X - X$ is totally disconnected, then $\alpha X \leq FX$.

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In [11], the author introduced

Definition 1. Let vX be the Hewitt realcompactification of X, put $K_X = \mathrm{cl}_{\beta X}(vX - X)$ and $\delta X = \beta X - (K_X - X)$. A set $H \subseteq X$ is *hard* in X if H is closed in $X \cup K_X$.

Notice that $X \cup K_X = vX \cup K_X$ is realcompact by [3, 8.16] and that $X \cap K_X$ is just the set of points at which X fails to be locally realcompact. Clearly a set H is hard in X if and only if it is the restriction of a compact set of δX to X. Every compact set of X is hard and every hard set is closed and realcompact, though neither converse is, in general, true.

Definition 2. A space is *rimhard* if it has a basis of open sets with hard frontiers.

Easily, both rimcompact spaces and locally realcompact (= locally hard [11]) spaces are rimhard. Examples are given in §4 of a rimcompact but not locally realcompact space, and of a locally realcompact but not rimcompact space.

Lemma 1. *Let X be dense in T and for each $U \subseteq T$, let $\partial_T U$ be the frontier of U in T. Then for every open $U \subseteq T$, $\partial_X(X \cap U) = X \cap \partial_T U$.*

Theorem 1. *X is rimhard if and only if δX is rimcompact.*

Proof. (If). Let A be an open set of X, $p \in A$ and A_0 open in δX such that $A_0 \cap X = A$. Then there is an open set G_0 in δX with $p \in G_0 \subseteq A_0$ such that $\partial_{\delta X} G_0$ is compact. Let $G = X \cap G_0$. Then $p \in G \subseteq A$ and $\partial_X G = X \cap \partial_{\delta X} G_0$, which is hard.

(Only if). Let U be a nonempty open set in δX and G be open in βX such that $G \cap \delta X = U$. Either (1) $G \not\subseteq K_X$ or (2) $G \subseteq K_X$. (Notice that $G \subseteq K_X$ if and only if $U \subseteq K_X$.)

Case 1. $G \not\subseteq K_X$. Since K_X is compact, $G - K_X$ is a non-$\emptyset$, βX-open subset of U. Let $p \in G - K_X$. Since βX is regular, there is a βX-open V such that $p \in V \subseteq \mathrm{cl}_{\beta X} V \subseteq G - K_X$. By the lemma, $\partial_{\beta X} V = \partial_{\delta X} V$, which is thus compact.

Case 2. $U \subseteq K_X \cap X$. Thus $U = U \cap X$ is open in X, so there is an X-open $A \subseteq U$ with $\partial_X A$ hard. Let H be open in βX such that $H \cap X = A$, and let $T = H \cap G$. Then

$$\delta X \cap T = H \cap (\delta X \cap G) = H \cap U \subseteq U \subseteq X.$$

Thus $H \cap U \subseteq H \cap X = A$. So $T \cap \delta X = A$ and A is open in δX. Now $A \subseteq X \cap K_X \subseteq K_X$. Let $p \in \delta X - K_X$. Then p and K_X are disjoint closed sets in

βX, so there exist βX-open disjoint sets S and M with $p \in S$ and $K_X \subseteq M$. Then $p \in S \cap \delta X$, open in δX and disjoint from K_X. Whence $(S \cap \delta X) \cap A = \emptyset$ and $p \notin \partial_{\delta X} A$. Thus $\partial_{\delta X} A \subseteq X \cap K_X$.

By Lemma 1, $\partial_X A = X \cap \partial_{\delta X} A = \partial_{\delta X} A$. Therefore

$$\partial_X A = \mathrm{cl}_{X \cup K_X} \partial_X A \subseteq K_X \subseteq X \cup K_X$$

and K_X is compact. Therefore $\partial_{\delta X} A$ is compact and δX is rimcompact.

2. **Close and tight realcompactifications.** In [6], we considered the partially ordered family of tight realcompactifications (i.e. containing no proper realcompactifications) of X and its algebraic influence on the topological structure of $\upsilon X - X$. The compactifications of δX give rise in a natural way to another family of realcompactifications of X, whose algebraic structure can shed light on the topology of $K_X - X$. In order to investigate this second family and its relations to the tight realcompactifications, we shall need [11],

Proposition 1. *Let X be realcompact and f be a quotient map from X onto Y. Let $M = \{y \in Y: |f^{\leftarrow}(y)| > 1\}$. Then Y is realcompact if and only if $\mathrm{cl}_Y M$ is hard in Y.*

Lemma 2. *If $\alpha(\delta X)$ is a compactification of δX, then $\alpha(\delta X) - (\delta X - X)$ is a realcompactification of X.*

Proof. Let $\alpha(\delta X) - (\delta X - X) = S$. Let f be the quotient map from $\beta(\delta X) = \beta X$ onto $\alpha(\delta X)$ which fixes δX pointwise. By [3, 6.12], we have $f[\beta X - \delta X] = f[K_X - X] = S - X$. Let g be the restriction of f to realcompact $X \cup K_X$, so g is a quotient map onto S. But $M = \{y \in S: |g^{\leftarrow}(y)| \geq 1\} \subseteq S - X \subseteq f[K_X]$ which is compact. Therefore $\mathrm{cl}_S M$ is hard in S and by Proposition 1, S is realcompact. Clearly X is dense in S.

Definition 3. A realcompactification S of X is called *close* iff $S = \alpha(\delta X) - (\delta X - X)$ for some compactification $\alpha(\delta X)$ of SX. Since $\alpha(\delta X)$ is both a compactification of S and a quotient of βX which leaves δX pointwise fixed, it follows that $\alpha(\delta X) = \beta S$. Denote the partially ordered family of close realcompactifications as $P_\delta(X)$.

It is clear that $P_\delta(X)$ is isomorphic to $\mathcal{K}(\delta X)$, the compactifications of δX. This allows us to make certain immediate statements about $P_\delta(X)$; for example, by [6, Lemma 2.10], X is locally realcompact and not realcompact (hereafter stated as X is l.r.) if and only if δX is locally compact and not compact. In this case the structure of $\mathcal{K}(\delta X)$ characterizes $\beta(\delta X) -$

$\delta X = K_X$ [7]. So for an l.r. space X, the algebraic structure of $P_\delta(X)$ determines the topology of K_X. In particular, the close realcompactification of X corresponding to the Alexandrov compactification of δX will be $^*X$, the maximal one-point realcompactification of X. This was constructed in [5, Theorem 4.1] by collapsing K_X (which is here disjoint from X) to a single point.

From [6, Lemma 3.4], we have

Proposition 2. *Let T be a tight realcompactification of X and let $f\colon \beta X \to \beta T$ be the continuous map fixing X pointwise. The following are equivalent:*

(a) *the restriction of f to $X \cup K_X$ is perfect (i.e. a closed map with compact fibers);*

(b) *the restriction of f to $X \cup K_X$ is quotient;*

(c) *the restriction of f to δX is a homeomorphism;*

Moreover if X is l.r., these are equivalent to

(d) $^*X \leq T$.

Definition 4. Let T be a realcompactification of X. Then $T \in P^*(x)$ iff T is the tight realcompactification of X contained in some compactification $\alpha(\delta X)$ of δX.

Lemma 3. *Let T be a tight realcompactification of X. Then $T \in P^*(X)$ iff $\beta T \in \mathcal{K}(\delta X)$.*

Proof. (If) is trivial. Conversely, there is some compactification $\alpha(\delta X)$ of δX which contains T. Since T is dense in $\alpha(\delta X)$, there is a continuous map g from βT onto $\alpha(\delta X)$ preserving T pointwise. But βT is a compactification of X, so there is a continuous map f from βX onto βT fixing X pointwise. Thus $g \circ f$ is a continuous map from βX onto $\alpha(\delta X)$ which fixes at least X pointwise. But $\beta X = \beta(\delta X)$, so there is a continuous map h from βX onto $\alpha(\delta X)$ which fixes δX pointwise. Since h and $g \circ f$ agree on the dense subset X, they must agree everywhere. In particular f must fix δX pointwise, i.e. βT is a compactification of δX.

Corollary 1. *Let T be a tight realcompactification of X. Then $T \in P^*(X)$ iff T satisfies any one part (hence all) of Proposition 2.*

Corollary 2. *If $T \in P^*(X)$, then $X \cup \mathrm{cl}_{\beta T}(T - X) \in P_\delta(X)$.*

Proof. βT is a compactification of δX and $X \cup \mathrm{cl}_{\beta T}(T - X) = \beta T - (\delta X - X)$. The result now follows from Lemma 2.

$P^*(X)$ has been discussed in [6, 5(A)], where it was observed that if X is l.r. and υX is Lindelöf, its algebraic structure characterizes the topology of $\upsilon X - X$. By [3, 8.9], every close realcompactification contains a tight realcompactification; but as υX and $X \cup K_X$ show, they are generally distinct.

As a final remark, we see that while the partial order of $P_\delta(X)$ is that of the compactifications of δX, the partial order of $P^*(X)$ can be obtained from that of the compactifications of X as in [6, Lemma 3.2].

Proposition 3. *Let T_1 and T_2 be in $P^*(X)$. Then $T_1 \leq T_2$ in $P^*(X)$ if and only if $\beta T_1 \leq \beta T_2$ in $\mathcal{K}(X)$.*

3. Realcompactifications with 0-dimensional outgrowth.

Definition 5. Suppose B is a proper subset of space A. Then A is *extensively disconnected outside* B if $\mathrm{cl}_{\beta A}(A - B) - B$ is totally disconnected.

In particular, let $T \in P^*(X)$ and S be its generated close realcompactification as in Corollary 2 of Lemma 3. Clearly T is extensively disconnected outside X if and only if $S - X$ is totally disconnected. Recollect that a space X is strongly 0-dimensional [2, Chapter 6, §2] if any two completely separated sets of X are separated by a partition. Every strongly 0-dimensional space is 0-dimensional, and if X is Lindelöf, the two are equivalent.

Lemma 4. *Let B be a proper subset of space A. If $A - B$ is C^*-embedded in A and is strongly 0-dimensional, then A is extensively disconnected outside B.*

Proof. By [3, 6.9(a)], $\beta(A - B) = \mathrm{cl}_{\beta A}(A - B)$. By [2, Theorem 6.8], $\beta(A - B)$ is strongly 0-dimensional if and only if $A - B$ is.

Theorem 2. *Let X be rimhard. Then there is a close realcompactification S_0 of X such that $S_0 - X$ is 0-dimensional. Moreover, if S is in $P_\delta(X)$ and $S - X$ is totally disconnected, then $S \leq S_0$.*

Proof. By Theorem 1, δX is rimcompact and hence admits the Freudenthal compactification $F(\delta X)$. Let S_0 be the close realcompactification of X such that $S_0 - X = F(\delta X) - \delta X$. Then $S_0 - X$ is 0-dimensional, and since $P_\delta(X)$ is isomorphic to $\mathcal{K}(\delta X)$, the maximality condition follows immediately from the corresponding statement for $F(\delta X)$.

Corollary 1. *Let X be rimhard. There is a tight realcompactification T_0 in $P^*(X)$ such that $T_0 - X$ is 0-dimensional. Moreover if $T \in P^*(X)$ is extensively disconnected outside X, then $T \leq T_0$.*

Proof. Let T_0 be the tight realcompactification of X contained in S_0. Since 0-dimensionality is hereditary, $T_0 - X$ is 0-dimensional. Suppose $T \in P^*(X)$ is extensively disconnected outside X and that $S_T = \beta T - (\delta X - X)$. Then since $T \subseteq S_T \subseteq \beta T$, we have $\beta T = \beta S_T \leq \beta S_0$. Since $T_0 \subseteq S_0$, we have $\beta S_0 \leq \beta T_0$. Hence $\beta T \leq \beta T_0$ and by Proposition 3, $T \leq T_0$.

Corollary 2. *Let X be rimhard and $\upsilon X - X$ be closed in $\beta X - X$. Let T_0 be the tight realcompactification of Corollary 1. If $T \in P^*(X)$ has $T - X$ totally disconnected, then $T \leq T_0$.*

Corollary 3. *Let X be l.r. and $T \in P^*(X)$ be Lindelöf. If $T - X$ is 0-dimensional, then $T \leq T_0$.*

Proof. $T - X$ is a closed subspace of Lindelöf (hence normal) T, so $T - X$ is C^*-embedded in T. Since a Lindelöf 0-dimensional space is strongly 0-dimensional, the result now follows from Lemma 4.

Notice that if υX is Lindelöf, then T is Lindelöf for every T in $P^*(X)$.

4. **Examples.** A. Two rimhard spaces.

(i) A rimcompact space which is not locally realcompact. Let N be the space of positive integers and W the space of ordinals less than the first uncountable ordinal ω_1. Let $W^* = W \cup \{\omega_1\}$ be its compactification, and set $X = (W^* \times \beta N) - (\{\omega_1\} \times N)$. This X is pseudocompact, so $\upsilon X = \beta X = W^* \times \beta N$. Since X has a countable compactification, by [10, Theorem 1, Corollary 2], it is rimcompact. But the set of points at which X is not locally realcompact is $X \cap \mathrm{cl}_{\beta X}(\upsilon X - X)$, a copy of $\beta N - N$.

(ii) A locally realcompact space which is not rimcompact. Let Y be the subspace of the plane formed by the origin and all rays from the origin of slope $1/n$ for each positive integer n. Then Y is Lindelöf, so realcompact, yet the frontier of an open disk about the origin hits Y in a Cauchy sequence which fails to converge in Y. Thus Y is not rimcompact.

Indeed, if we let X be the free union of Y and W, then X is neither rimcompact nor realcompact. But X admits a one point realcompactification $Y \cup W^*$, so X is l.r.

B. A nonrimhard space having a maximal close realcompactification with 0-dimensional outgrowth. Let W be the countable ordinals and $W^* = W \cup \{\omega_1\}$ be its one point compactification. Let Y be a nonrimcompact space having a compactification αX such that $\alpha Y - Y$ is 0-dimensional; such a space can be found in [4, VII, 25]. Define $X = [W^* \times \alpha Y] - [\{\omega_1\} \times (\alpha Y - Y)]$. Then $\beta X - X = \{\omega_1\} \times (\alpha Y - Y)$ is 0-dimensional and the residue of X (i.e., the set of points at which X is not locally compact) is $R(X) = \{\omega_1\} \times [Y \cap \mathrm{cl}_{\alpha Y}(\alpha Y - Y)]$. Since Y cannot be locally compact, $R(X)$ is a nonempty closed subset of X and since rimcompactness is closed-hereditary, X is not rimcompact. But X is pseudocompact, so $X = \delta X$ and by Theorem 1, X is not rimhard. Yet $\beta X - X = \upsilon X - X = \mathrm{cl}_{\beta X}(\upsilon X - X) - X$, so βX is the maximal close realcompactification of X. This example answers a question of the referee as to the converse of the first part of Theorem 2.

The author wishes to thank the referee for his valuable criticisms and suggestions. In particular, these led to a significant improvement in what is now Lemma 3 and its corollaries, and to the uncovering of an error in the original version of what is now Corollary 1 to Theorem 2. The author had claimed that the tight realcompactification T_0 of that result was itself extensively disconnected outside X. The proof of this was certainly in error, and although no counterexample has come to hand, the author now conjectures that the claim itself is false.

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