A CLASS OF L^p-BOUNDED PSEUDO-DIFFERENTIAL OPERATORS

REINHARD ILLNER

ABSTRACT. Pseudo-differential operators with symbol $p(x, \xi, y) \in S^\mu_{\rho, \delta, \epsilon}$, $\mu \leq (\rho - 1)(n + 1)$, are proven to be generally L^p-bounded for $1 < p < \infty$.

Introduction. Previously L^2-boundedness of pseudo-differential operators with symbol $p(x, \xi, y) \in S^\mu_{\rho, \delta, \epsilon}$ was shown by Hörmander [3] and Calderón and Vaillancourt [1], if $\mu, \rho, \delta, \epsilon$ satisfy suitable conditions. In older theorems by Mikhlin (see [4, pp.232 ff.]) and Hörmander [2, pp. 120–123] the general L^p-boundedness for $\mathbf{R}^n$ Fourier-multipliers with symbol $a(\xi)$ is proven, where $a(\xi)$ satisfies conditions closely related to those on $p(x, \xi, y)$ in [1] and [3]. In [5], Muramatu gave a generalization of Calderón's result and proved general L^p-boundedness for a class of pseudo-differential operators, imposing additional conditions on the Fourier transform of $p(x, \xi, y)$. It is the purpose of this paper to replace these conditions by conditions on the symbol itself, such that L^p-boundedness of the operator is still true for $1 < p < \infty$.

The problem was pointed out to the author by Professor H. O. Cordes, who gave more valuable advice.

Muramatu's notation is widely used, and parts of the proof of the main theorem are identical to the proof of his result in [5]. Other methods used in this paper are similar to Hörmander's in [2].

Notation. $|x|$ denotes the Euclidean norm of $x \in \mathbf{R}^n$, and $|x|_\infty = \sup |x_j|$, where $j = 1, 2, \ldots, n$. If $\alpha = (\alpha_1, \ldots, \alpha_n)$ with $\alpha_i = 0, 1, \ldots$, then $D^\alpha_\xi = \partial^{\alpha_1 + \cdots + \alpha_n} / \partial \xi_1^{\alpha_1} \ldots \partial \xi_n^{\alpha_n}$ and similarly D^α_x, D^α_y for $x, y, \xi \in \mathbf{R}^n$. $|\alpha| = \alpha_1 + \cdots + \alpha_n$. $\mathcal{S}(\mathbf{R}^n)$ is the space of C-valued rapidly decreasing functions. $m(M)$ denotes the measure of the set $M \subset \mathbf{R}^n$. We say that $p(x, \xi, y)$ has compact support in ξ if $\{\xi | p(x, \xi, y) \neq 0\}$ is for all x, y contained in a minimal fixed compact set, and we denote this set by $\text{supp}_\xi p$.

Definition. A C-valued symbol $p(x, \xi, y)$ of $(x, \xi, y) \in \mathbf{R}^n \times \mathbf{R}^n \times \mathbf{R}^n$ belongs to $S^\mu_{\rho, \delta, \epsilon}(\mathbf{R}^{3n}, \mathbf{C})$ if

(1) $$|D_x^\alpha D_\xi^\beta p(x, \xi, y)| \le C(1 + |\xi|)^{\mu + \delta|\alpha| - \rho|\beta|},$$

(2) $$|D_y^\gamma D_\xi^\beta p(x, \xi, y)| \le C(1 + |\xi|)^{\mu + \epsilon|\gamma| - \rho|\beta|}$$

for any multi-index α, β, γ, where $0 \le \rho, \delta, \epsilon \le 1$.

We define a pseudo-differential operator T on S by

(3) $$Tf(x) = (2\pi)^{-n} \int e^{ix\xi} d\xi \int p(x, \xi, y)f(y)e^{-i\xi y} dy.$$

Tf is well defined and belongs to S.

Theorem 1 (Calderón-Vaillancourt [1]). *Let $0 \le \delta, \epsilon < 1$, $0 \le \rho \le 1$, and $-2\mu \ge n\{\max(\delta, \rho) + \max(\epsilon, \rho)\} - 2n\rho$. Let $p(x, \xi, y)$ be a symbol satisfying (1) and (2) with these $\rho, \delta, \epsilon, \mu$ for $0 \le |\beta| \le 2m$, $0 \le |\alpha| \le 2m_1$, $0 \le |\gamma| \le 2m_2$, where m, m_1, m_2 are the least integers such that $2m \ge n + 2$, $m_1(1 - \delta') > 5n/4$, $m_2(1 - \epsilon') > 5n/4$, $\rho' = \min(\rho, \max(\delta, \epsilon))$, $\delta' = \max(\delta, \rho')$, $\epsilon' = \max(\epsilon, \rho')$. Then*

(a) $\|Tf\|_{L^2} \le C\|p\| \cdot \|f\|_{L^2}$, *where C depends on $\delta, \epsilon, \rho, n$ only. $\|p\|$ denotes the least value for which (1) and (2) hold if we restrict $|\alpha|, |\beta|, |\gamma|$ like above.*

(b) *If p and p_j satisfy the conditions of (a), p_j has compact support in ξ, $\|p_j\|$ is bounded and for all $\psi(\xi) \in C_0^\infty$ $\|(p_j - p)\psi\| \to 0$ as $j \to \infty$, the operator T_j associated to p_j as in (3) converges strongly to a limit T that depends only on p, and $\|T\| \le C\|p\|$, with the constant C of (a).*

Proof. See Calderón-Vaillancourt [1].

Theorem 2 (Marcinkiewicz interpolation theorem). *Let $1 < q \le \infty$ and let T be a subadditive mapping from $L^1(\mathbf{R}^n) + L^q(\mathbf{R}^n)$ into the space of measurable functions on $\mathbf{R}^n$. If for all $\lambda > 0$*

(4) $m\{x||Tf(x)| > \lambda\} \le C_1/\lambda \cdot \|f\|_{L^1}$ *and*

(5) $m\{x||Tf(x)| > \lambda\} \le (C_2/\lambda \cdot \|f\|_{L^q})^q$ *(when $q = \infty$ we assume $\|Tf\|_\infty \le C_2\|f\|_\infty$), we have for all $1 < p < q$,*

$$\|Tf\|_{L^p} \le C_p\|f\|_{L^p},$$

where C_p depends only on C_1, C_2, p and q.

Proof. See, e.g., E. M. Stein [6, pp. 272–274].

(4) and (5) are called weak L^1-boundedness and weak L^q-boundedness of T respectively. Next we are going to state our main result.

Theorem 3. *Assume $1 < p < \infty$. Let the function $p(x, \xi, y)$ satisfy the conditions of Theorem 1 with $\mu \le (\rho - 1)(n + 1)$. Then the pseudo-differential*

operator

$$(Tf)(x) = (2\pi)^{-n} \int p(x, \xi, y) f(y) e^{i(x-y)\xi} \, dy \, d\xi$$

is bounded from $L^p(\mathbf{R}^n)$ to $L^p(\mathbf{R}^n)$.

Proof. From (1) and (2) with $\mu \leq (\rho - 1)(n + 1)$, the reader will have no difficulty verifying that there is a constant B such that

(a) $\int_{\frac{1}{2}R \leq |\xi| \leq 2R} |R^{|\alpha|} D_\xi^\alpha p(x, \xi, y)| \, d\xi \leq BR^n$

$$\text{for all } |\alpha| \leq n + 1 \text{ and all } R \geq 1,$$

(b) $\int_{\frac{1}{2}R \leq |\xi| \leq 2R} |R^{|\alpha|} D_\xi^\alpha (\partial/\partial y_l) p(x, \xi, y)| \, d\xi \leq BR^{n+1}$, and

$$\int_{\frac{1}{2}R \leq |\xi| \leq 2R} |R^{|\alpha|} D_\xi^\alpha (\partial/\partial x_l) p(x, \xi, y)| \, d\xi \leq BR^{n+1}$$

$$\text{for all } |\alpha| \leq n + 1, \ R \geq 1, \ l = 1, \ldots, n.$$

For the main part of the proof, we shall assume that $\mathrm{supp}_\xi \, p$ is compact. At the end we give an argument that enables us to abandon this restriction on p and thus yields the theorem's statement in the desired generality. In the whole proof, C will denote constants depending on n and the so far introduced constants $(B, \|p\|, C)$, but C may have different values in different formulas.

So, assume that p has compact support in ξ. In view of Theorem 1, part (a), and Theorem 2, it suffices to prove weak L^1-boundedness for T. Choose a fixed $v \in C_0^\infty(\mathbf{R}^n)$ such that $v(\xi) \in [0, 1]$ for all ξ, and

$$v(\xi) = \begin{cases} 1 & \text{if } |\xi| \leq 1, \\ 0 & \text{if } |\xi| \geq 2, \end{cases}$$

and let $w = 1 - v$. Then obviously $p = pv + pw$ and $\mathrm{supp}_\xi (pv) \subset \{\xi; |\xi| \leq 2\}$.

Lemma 1. *The pseudo-differential operator with symbol pv is L^1-bounded and so in particular weakly L^1-bounded.*

Proof. Let $\kappa = [n/2] + 1$.

$$\int |Tu(x)| \, dx = (2\pi)^{-n} \int \left| \int \int pv(x, \xi, y) e^{i(x-y)\xi} \, d\xi u(y) \, dy \right| dx$$

$$= (2\pi)^{-n} \int \left| \int \int \int \frac{1}{(1 + |x - y|^2)^\kappa} (1 - \Delta_\xi)^\kappa (pv)(x, \xi, y) e^{i(x-y)\xi} \, d\xi u(y) \, dy \right| dx$$

$$\leq (2\pi)^{-n} \int \int \frac{1}{(1 + |x - y|^2)^\kappa} \int |(1 - \Delta_\xi)^\kappa (pv)(x, \xi, y)| \, d\xi |u(y)| \, dy \, dx \leq C|u|_1$$

by the Fubini theorem, by $2\kappa \geq n + 1$ and because pv is bounded and has compact support in ξ, that is not dependent on x, y. □

It remains to prove that the operator with symbol pw is weakly L^1-bounded. Note that $pw = 0$ if $|\xi| \leq 1$. As the operator with symbol pw satisfies the conditions of Theorem 1 and (a) and (b) (eventually with different constants, that, however, are dependent on p and the fixed w only), we shall henceforth assume $pw = p$, i.e. p satisfies the conditions of the theorem and $p(x, \xi, y) = 0$ if $|\xi| \leq 1$.

By Calderón-Zygmund's theorem [6, p. 17] there is for $f \in L^1$ and $\lambda > 0$ a decomposition of $\mathbf{R}^n$ so that $\mathbf{R}^n = F \cup \Omega$, $F \cap \Omega = \varnothing$, $|f(x)| \leq \lambda$ a.e. on F, Ω is the union of cubes $\Omega = \bigcup_k Q_k$, whose interiors are disjoint, and

$$\lambda m(Q_k) \leq \|f\|_{L^1(Q_k)} \leq 2^n \cdot \lambda \cdot m(Q_k).$$

Let Q_j° denote the interior of Q_j. Furthermore, let

$$f_0(x) = \begin{cases} f(x), & x \in F, \\ \dfrac{1}{m(Q_j)} \displaystyle\int_{Q_j} f(y)\, dy, & x \in Q_j^\circ, \end{cases}$$

and let $g(x) = f(x) - f_0(x)$. By straightforward computation, we find

$$\|f_0(x)\|_{L^2}^2 \leq \lambda(1 + 2^{2n})\|f\|_{L^1},$$

and from this we get with Theorem 1

(6) $$m\{x;\ |Tf_0(x)| > \lambda\} \leq (C/\lambda)\|f\|_{L^1}.$$

Now, after Tf_0, we are going to estimate Tg.

Lemma 2. *There is a function* $\varphi \in C_0^\infty$ *with support in* $\{\frac{1}{2} < |\xi| < 2\}$ *such that* $\sum_{j=0}^\infty \varphi(2^{-j}\xi) = 1$ *if* $|\xi| \geq 1$.

Proof. Let $\phi \geq 0$ be in C_0^∞ with support in $\{\frac{1}{2} < |\xi| < 2\}$ and $\phi(\xi) > 0$ if $1/\sqrt{2} < |\xi| < \sqrt{2}$. Set

$$\varphi(\xi) = \phi(\xi) \Big/ \sum_{j=-\infty}^{\infty} \phi(2^{-j}\xi).$$

Then the denominator is never 0 for $\xi \neq 0$, and so $\varphi \in C_0^\infty$. If $|\xi| > 1$, we have $|2^k\xi| > 2$ $(k > 0)$, and this proves the lemma. □

Let $p_j(x, \xi, y) = p(x, \xi, y) \cdot \varphi(2^{-j}\xi)$ and notice that $\mathrm{supp}_\xi\, p_j \subset \{2^{j-1} <$

$|\xi| < 2^{j+1}\}$ for $j = 0, 1, 2, \ldots$. Let Q be an arbitrary, nonempty cube with sides parallel to the axes. Let $2r$ be the length of the side of Q, $s = \sqrt{n}\,r$, x_0 the center of Q, $B = \{x; |x - x_0| \leq 2s\}$. Furthermore, let u be a function with support in Q such that $\int u = 0$. We shall show

$$(7) \qquad \int_{|x-x_0| \geq 2s} \left| \int p(x, \xi, y) u(y) e^{i(x-y)\xi}\, dy\, d\xi \right| dx \leq C \int |u|\, dx$$

with a constant C dependent neither on s nor on $\mathrm{supp}_{\xi}\, p$.

To do so, use the Leibniz' formula to get

$$D_{\xi}^{\alpha} p_j(x, \xi, y) = \sum_{\beta + \gamma = \alpha} 2^{-j|\beta|}(D_{\xi}^{\gamma} p(x, \xi, y))(D_{\varphi}^{\beta})(2^{-j}\xi),$$

and since the derivatives of φ are bounded, we get from (a) with $R = 2^j$:

$$(8) \qquad \int_{2^{j-1} \leq |\xi| \leq 2^{j+1}} |2^{j|\alpha|} D_{\xi}^{\alpha} p_j(x, \xi, y)|\, d\xi \leq CB 2^{nj}.$$

The next estimate has to be done differently for n odd and n even.

(A) *n odd*. Pointwise in x, we use partial integration in ξ in the inner integral to compute

$$\int_{|x-y| \geq s} (1 + 2^{j(n+1)}|x - y|_{\infty}^{n+1}) \left| \int p_j(x, \xi, y) e^{i(x-y)\xi}\, d\xi \right|^2 dx$$

$$(9) \qquad = \int_{|x-y| \geq s} \frac{1}{|x - y|_{\infty}^{n+1}} \left| \int (D_{\xi}^{\alpha} p_j(x, \xi, y) e^{i(x-y)\xi} \right.$$

$$\left. + 2^{j((n+1)/2)} D_{\xi}^{\alpha'} p_j(x, \xi, y) e^{i(x-y)\xi}) \, d\xi \right|^2 dx,$$

where $|\alpha| = (n + 1)/2$, $|\alpha'| = n + 1$, and $\alpha = (0, 0, \ldots, 0, |\alpha|, 0, \ldots, 0)$, $\alpha' = (0, \ldots, |\alpha'|, \ldots, 0)$, the place of $|\alpha|$, $|\alpha'|$ depending on x. But (9) is less than

$$(C/s)(2^{-j((n+1)/2)} CB 2^{nj})^2 = (C/s)(B 2^{j(n-1)/2})^2$$

by the equivalence of $|\,\,|$ and $|\,\,|_{\infty}$ and by (8). The Cauchy-Schwarz inequality now yields

$$\int_{|x-y| \geq s} \left| \int p_j(x, \xi, y) e^{i(x-y)\xi}\, d\xi \right| dx$$

$$(10) \qquad \leq \frac{CB \cdot 2^{j(n-1)/2}}{\sqrt{s}} \left(\int \frac{dx}{1 + 2^{j(n+1)}|x - y|_{\infty}^{n+1}} \right)^{1/2} = CB(2^j s)^{-1/2}.$$

The above estimate for $\int_{|x-y|\geq s} |\int p_j(x, \xi, y)e^{i(x-y)\xi}\, d\xi|\, dx$ is good if $2^j s$ is big. For small $2^j s$ we use another estimate. For this, observe that because $\int u = 0$, we can, to obtain (7), replace $\int_{|x-y|\geq s} |\int p_j(x, \xi, y)\, e^{i(x-y)\xi}\, d\xi|\, dx$ by

$$K_j(x_0, y, s) = \int_{|x-x_0|\geq 2s} \left| \int p_j(x, \xi, y)e^{i(x-y)\xi}\, d\xi \right.$$

$$\left. - \int p_j(x, \xi, x_0)e^{i(x-x_0)\xi}\, d\xi \right|\, dx.$$

Because $\operatorname{supp} u \subset Q$, $|x - x_0| \geq 2s$ implies $|x - y| \geq s$, and so we find like above

(11) $$K_j(x_0, y, s) \leq CB(2^j s)^{-1/2}.$$

Now let $y(t) = y + t(x_0 - y)$, $t \in [0, 1]$. Then

$K_j(x_0, y, s)$

$$= \int_{|x-x_0|\geq 2s} \left| \int_\xi \int_0^1 \nabla_y(p_j(x, \xi, y(t)))e^{i(x-y(t))\xi})(x_0 - y)\, dt\, d\xi \right|\, dx$$

$$\leq \int_{t=0}^1 \int_{|x-x_0|\geq 2s} \left| \sum_{m=1}^n \int D_{y_m}(p_j(x, \xi, y(t))e^{i(x-y(t))\xi})(x_{0m} - y_m)\, d\xi \right|\, dx\, dt$$

$$\leq s \cdot \sum_{m=1}^n \int_{t=0}^1 \int_{|x-x_0|\geq 2s} \left| \int D_{y_m}(p_j(x, \xi, y(t))e^{i(x-y(t))\xi})\, d\xi \right|\, dx\, dt,$$

because $|x_{0m} - y_m| \leq s$. The integral over x is now estimated exactly like above. Observing that $|\xi| \leq 2^{j+1}$ and using just (b) and the ordinary differentiation rules, we find

$$\sum_{m=1}^n \int_{t=0}^1 \int_{|x-x_0|\geq 2s} \left| \int_\xi D_{y_m}(p_j(x, \xi, y(t))e^{i(x-y(t))\xi})\, d\xi \right|\, dx\, dt$$

$$\leq (CB/\sqrt{s})2^{j/2},$$

i.e.

(12) $$K_j(x_0, y, s) \leq CB(2^j s)^{1/2},$$

which is a good estimate whenever $2^j s$ is small.

(B) *n even.* We estimate again $K_j(x_0, y, s)$, obtaining the same result. The estimate works in almost the same way, except: (1) The function $1 +$

$2^{j(n+1)}|x - y|_\infty^{n+1}$ has to be replaced by $f = |x - y|_\infty (2^j + 2^{j(n+1)}|x - y|_\infty^n)$.
(2) We obtain $1/|x - y|_\infty^{n+2}$ in the integral over x by differentiating $(n/2 + 1)$ times in the inner integral. One power of $|x - y|_\infty$ cancels the factor $|x - y|_\infty$ in f. We end up with the same result. So we have for all dimensions n

$$(13) \qquad K_j(x_0, y, s) \le CB \cdot \min\{(2^j s)^{-1/2}, (2^j s)^{+1/2}\},$$

and the estimate is valid for all x_0 and all functions u satisfying the stated conditions. Let

$$q_j(x, y) = \int_\xi p_j(x, \xi, y) e^{i(x-y)\xi} \, d\xi,$$

and let $Q_N = \sum_{j=0}^N q_j$. Using the triangle inequality and (13), we find

$$(14) \qquad \begin{aligned} \int_{|x-x_0| \ge 2s} &|Q_N(x, y) - Q_N(x, x_0)| \, dx \\ &\le CB \sum_{j=0}^\infty \min\{(2^j s)^{-1/2}, (2^j s)^{1/2}\} \le C, \end{aligned}$$

since the sum is a bounded function in s. The constant does not depend on N. If we set $P_N = \sum_{j=0}^N p_j$, we get $p = P_N$ for all $N \ge N_0$, where N_0 depends on the support of p. Notice that P_N yields the operator kernel Q_N.

Lemma 3. *Let $h(x, y)$ be a kernel function for an integral operator H defined by $Hu(x) = \int h(x, y)u(y) \, dy$, define $h_y(x) = h(x, y)$ and assume $h_y \in L^1$ for all y. Then*

$$\|Hu\|_{L^1} \le \left(\operatorname*{ess\,sup}_y \{\|h_y\|_1\} \right) \cdot \|u\|_{L^1}.$$

The proof is a simple application of the Fubini theorem. $\square$

As an immediate consequence of the lemma and estimate (14), we have

$$(7) \qquad \begin{aligned} \int_{|x-x_0| \ge 2s} \Bigg| \int_{|y-x_0| \le s} \int (P_N(x, \xi, y) e^{i(x-y)\xi} \\ - P_N(x, \xi, x_0) e^{i(x-x_0)\xi}) u(y) \, d\xi dy \Bigg| \, dx \le C \int |u| \, dy. \end{aligned}$$

Now remember the definition of g and Q_k. Let x^k be the center of Q_k, $2b_k$ the length of the side of Q_k, $r_k = \sqrt{n} \, b_k$, $B_k = \{x; |x - x^k| \le 2r_k\}$, $D' = \bigcup_k B_k$, $D = \mathbf{R}^n \setminus D'$,

$$g_k(x) = \begin{cases} g(x), & x \in Q_k', \\ 0, & \text{otherwise.} \end{cases}$$

Notice that $\int g_k = 0$ and that $\mathrm{supp}\, g_k \subset Q_k^{\circ}$. (7) implies, if $N \geq N_0$,

$$\int_D |Tg_k(x)|\, dx \leq C\|g_k\|_{L^1},$$

where the constant C is not dependent on k. So

$$\int_D |Tg(x)|\, dx \leq C \int |g(x)|\, dx,$$

and as $g = f - f_0$, we find

$$\int |g(x)|\, dx \leq C(\|f\|_{L^1} + 2^n \lambda m(\Omega)) \leq C \cdot (1 + 2^n)\|f\|_{L^1},$$

i.e.

$$\int_D |Tg(x)|\, dx \leq C \cdot \|f\|_{L^1}.$$

This implies $m(D \cap \{x|\ |Tg(x)| > \lambda\}) \leq (C/\lambda)\|f\|_{L^1}$. Since $m(D') \leq Cm(\Omega)$, we have

(15) $$m\{x|\,|Tg(x)| > \lambda\} \leq (C/\lambda)\|f\|_{L^1}.$$

So, because

$$m\{x;|Tf(x)| > 2\lambda\} \leq m\{x;|Tf_0(x)| > \lambda\} + m\{x;|Tg(x)| > \lambda\},$$

we get from (6) and (15) the weak L^1-boundedness for T. The strong L^p-boundedness for $1 < p \leq 2$ follows immediately from this and Theorem 1 (a), as an application of Theorem 2.

For $2 < p < \infty$, let $1/q = 1 - 1/p$. Then $1 < q < 2$. We have for $f, g \in \mathcal{S}(\mathbf{R}^n)$

$$\left|\int (Tf)(x)g(x)\, dx\right| = \left|\int (2\pi)^{-n} \int p(x,\ \xi,\ y)f(y)e^{i(x-y)\xi}\, dy\, d\xi\, g(x)\, dx\right|$$

$$= \left|\int (2\pi)^{-n} \int p(x,\ \xi,\ y)g(x)e^{i(x-y)\xi}\, dx\, d\xi\, f(y)\, dy\right|$$

$$\leq \|f\|_{L^p}\|T^*g\|_{L^q},$$

where T^* is the adjoint operator of T. If T is weakly L^1-bounded with constant C, so is T^*. As T^* is strongly L^2-bounded, we conclude as before

$$\left|\int (Tf)(x)g(x)\, dx\right| \leq \|f\|_{L^p}\|T^*g\|_{L^q} \leq C_q\|f\|_{L^p}\|g\|_{L^q},$$

and this proves general L^p-boundedness of T.

To abandon the restriction on $\mathrm{supp}_\xi\, p$, we can use methods identical to

those of the proof of Theorem 1(b), so the details are left to the reader. The statement of the theorem turns out to be valid for all p satisfying our conditions.

REFERENCES

1. A. P. Calderón and R. Vaillancourt, *A class of bounded pseudo-differential operators*, Proc. Nat. Acad. Sci. U. S. A. 69 (1972), 1185—1187. MR 45 #7532.

2. L. Hörmander, *Estimates for translation invariant operators in L^p spaces*, Acta Math. 104 (1960), 93—140. MR 22 #12389.

3. ———, *On the L^2 continuity of pseudo-differential operators*, Comm. Pure Appl. Math. 24 (1971), 529—535. MR 43 #6779.

4. S. G. Mihlin, *Multidimensional singular integrals and integral equations*, Fizmatgiz, Moscow, 1962; English transl., Pergamon Press, New York, 1965. MR 27 #5105; 32 #2866.

5. T. Muramatu, *On the boundedness of a class of operator-valued pseudo-differential operators in L^p-space*, Proc. Japan Acad. 49 (1973), 94—99.

6. E. M. Stein, *Singular integrals and differentiability properties of functions*, Princeton Math. Ser., no. 30, Princeton Univ. Press, Princeton, N. J., 1970. MR 44 #7280.

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF CALIFORNIA AT BERKELEY, BERKELEY, CALIFORNIA 94720

Current address: Institut für Angewandte Mathematik und Informatik der Universität Bonn, D-53 Bonn, Beringstrasse 4-6, Federal Republic of Germany