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ON PAIRS OF NONINTERSECTING FACES OF CELL COMPLEXES

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ABSTRACT. We show that, for all cell complexes whose underlying set is a manifold, M, an alternating sum of numbers of pairs of faces that do not intersect is a topological invariant. This is done by proving that it is a function of the Euler characteristic, χ, of M.

A cell complex [1, pp. 39–40] is a finite family, $\mathcal{C}$, of polytopes in $\mathbf{R}^n$ such that

(i) every face of a member of $\mathcal{C}$ is itself a member of $\mathcal{C}$;

(ii) the intersection of any two members of $\mathcal{C}$ is a face of each of them.

We shall call a polytope $P \in \mathcal{C}$ a face of $\mathcal{C}$. The number of i-dimensional faces of $\mathcal{C}$ will be denoted by f_i. The subset of $\mathbf{R}^n$ consisting of all the points of members of $\mathcal{C}$ will be denoted by $\operatorname{set}\mathcal{C}$. The boundary complex of a $(d+1)$-dimensional polytope, P, is the set of all faces of P of at most dimension d.

Let $\mathcal{C}$ be any cell complex such that $\operatorname{set}\mathcal{C} = M$ where M is a d-dimensional manifold. Then $\mathcal{C}$ will obey Euler's relation

$$\chi(M) = \sum_{i=0}^{d} (-1)^i f_i. \tag{1}$$

If $\mathcal{C}$ is the boundary complex of a $(d+1)$-polytope then M will be homeomorphic to the surface of a hypersphere and

$$\chi(M) = 1 + (-1)^d. \tag{2}$$

Let α_{ij} = the number of ordered pairs of faces of $\mathcal{C}$ of dimensions i and j that do not intersect. Then the α_{ij} are called incidence numbers of $\mathcal{C}$. Note that $\alpha_{ij} = \alpha_{ji}$. We are interested in the sum

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$$\psi(\mathcal{C}) = \sum_{i=0}^{d} \sum_{j=0}^{d} (-1)^{i+j}\alpha'_{ij}. \tag{3}$$

This and similar quantities have been investigated by Wu [2] and others.

Now we assign to each of the f_i i-dimensional faces of $\mathcal{C}$ a different number k ($k = 1, 2, 3, \ldots, f_i$). Then let $p_{ij}(k)$ = the number of j-dimensional faces of $\mathcal{C}$ intersecting with the i-dimensional face of $\mathcal{C}$ assigned the number k. Then

$$\alpha_{ij} = \sum_{k=1}^{f_i} (f_j - p_{ij}(k)) = f_i f_j - \sum_{k=1}^{f_i} p_{ij}(k). \tag{4}$$

We shall now prove

Theorem I. *If R is the i-dimensional face of $\mathcal{C}$ assigned the number k $(0 \leq i \leq d,\ 1 \leq k \leq f_i)$ then*

$$\sum_{j=0}^{d} (-1)^j p_{ij}(k) = (-1)^d. \tag{5}$$

Proof. Construct around R a figure Q' homeomorphic to the surface of a d-dimensional hypersphere and which contains exactly all the vertices of R within its interior. Then the intersection of the faces of $\mathcal{C}$ with Q' define a topological polytope, Q, of dimension d.

Let

$$r_j = \begin{cases} \#\text{ of } j\text{-dimensional faces of } R & \text{if } j < i, \\ 1 & \text{if } j = i, \\ 0 & \text{if } j > i, \end{cases}$$

$$q_j = \begin{cases} \#\text{ of } j\text{-dimensional faces of } Q & \text{if } 0 \leq j \leq d-1, \\ 0 & \text{if } j = -1. \end{cases}$$

Then, since every j-dimensional face of $\mathcal{C}$ emanating from (intersecting but not contained in) R intersects Q' in a $(j-1)$-dimensional face of Q, it is not hard to see that for all $0 \leq j \leq d$,

$$p_{ij}(k) = q_{j-1} + r_j. \tag{6}$$

Since Q is a d-dimensional topological polytope, and R is an i-dimensional one, by (2) we have

$$\sum_{j=0}^{d-1} (-1)^j q_j = 1 + (-1)^{d-1} = 1 - (-1)^d, \tag{7}$$

$$\sum_{j=0}^{d} (-1)^j r_j = \sum_{j=0}^{i-1} (-1)^j r_j + (-1)^i + \sum_{j=i+1}^{d} 0 = 1.$$

Then, by (6) and (7), (5) is true. Q.E.D.

Theorem II. $\psi(\mathcal{C}) = \chi^2(M) - \chi(M)$.

Proof. From (3) and (4),

$$\psi(\mathcal{C}) = \sum_{i=0}^{d} \sum_{j=0}^{d} (-1)^{i+j} f_i f_j - \sum_{i=0}^{d} \sum_{j=0}^{d} \sum_{k=1}^{f_i} (-1)^{i+j} p_{ij}(k). \tag{8}$$

From (1) we get

$$\chi^2(M) = \left(\sum_{i=0}^{d} (-1)^i f_i \right)^2 = \sum_{i=0}^{d} \sum_{j=0}^{d} (-1)^{i+j} f_i f_j.$$

And by (5) and (1)

$$\sum_{i=0}^{d} \sum_{j=0}^{d} \sum_{k=1}^{f_i} (-1)^{i+j} p_{ij}(k) = \sum_{i=0}^{d} (-1)^i \sum_{k=1}^{f_i} \sum_{j=0}^{d} (-1)^j p_{ij}(k) = (-1)^d \chi(M),$$

so that (8) becomes $\psi(\mathcal{C}) = \chi^2(M) - (-1)^d \chi(M)$. And, since $\chi(M) = 0$ whenever d is odd $\psi(\mathcal{C}) = \chi^2(M) - \chi(M)$. Q.E.D.

Corollary. *If $\mathcal{C}$ is the boundary complex of a $(d+1)$-dimensional polytope,* $\psi(\mathcal{C}) = 1 + (-1)^d$.

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