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APPROXIMATING ZEROS OF ACCRETIVE OPERATORS

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ABSTRACT. Let A be an m-accretive set in a reflexive Banach space E with a Gateaux differentiable norm. For positive r let J_r denote the resolvent of A. If the duality mapping of E is weakly sequentially continuous and 0 is in the range of A, then for each x in E the strong $\lim_{r\to\infty} J_r x$ exists and belongs to $A^{-1}(0)$. This is an extension to a Banach space setting of a result previously known only for Hilbert space.

Let H be a real Hilbert space and $U \subset H \times H$ a maximal monotone operator. For each positive r there is a unique y_r in H such that $0 \in y_r + rU(y_r)$. It is known [4] that if 0 belongs to the range of U, then the strong $\lim_{r\to\infty} y_r$ exists and is the point of $U^{-1}(0)$ closest to 0. It is our purpose in this note to extend this result to accretive operators in certain Banach spaces. According to [4], this leads to the possibility of calculating a zero of the given operator as the limit of an iteratively constructed sequence. Our method of proof is not a direct generalization of the Hilbert space proof. It works, however, only in a restricted class of Banach spaces. The question of whether our theorem is valid in other Banach spaces remains open.

Let E^* denote the dual of a real Banach space E. The duality mapping J from E into the family of nonempty subsets of E^* is defined by

$$J(x) = \{x^* \in E^*: (x, x^*) = \|x\|^2 \text{ and } \|x^*\| = \|x\|\}.$$

J is single-valued if and only if the norm of E is Gâteaux differentiable. If A is a subset of $E \times E$ and $x \in E$, we define

$$Ax = \{y \in E: [x, y] \in A\}$$

and set

$$D(A) = \{x \in E: Ax \neq \emptyset\}.$$

The range of A is defined by

$$R(A) = \bigcup \{Ax: x \in D(A)\}$$

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and its inverse by

$$A^{-1}y = \{x \in E:\ y \in Ax\}.$$

I will stand for the identity operator on E. The closure of a subset D of E will be denoted by $\mathrm{cl}(D)$. A mapping $T: D \to E$ is said to be nonexpansive if $\|Tx - Ty\| \le \|x - y\|$ for all x and y in D. In the sequel, $\to$ and $\rightharpoonup$ will denote strong and weak convergence respectively.

A subset A of $E \times E$ is called accretive [7] if for all $x_i \in D(A)$ and $y_i \in Ax_i$, $i = 1, 2$, there exists $j \in J(x_1 - x_2)$ such that $(y_1 - y_2, j) \ge 0$. Let D be a subset of E and A an accretive set (= accretive operator) with $D(A) \subset D$. A is said to be maximal accretive in D if there is no proper accretive extension B of A with $D(B) \subset D$. An accretive set A is maximal accretive if it is maximal accretive in E. It is m-accretive if $R(I + A) = E$. (It follows that $R(I + rA) = E$ for all positive r.) If $T: E \to E$ is nonexpansive, then $I - T$ is m-accretive. If A is m-accretive, then it is maximal accretive, but the converse is not true in general. If A is accretive one can define, for each $r > 0$, a nonexpansive single-valued mapping $J_r: R(I + rA) \to D(A)$ by $J_r = (I + rA)^{-1}$. It is called the resolvent of A. Conditions which imply that an m-accretive set is surjective can be found in [10].

The duality mapping J of a Banach space E with a Gâteaux differentiable norm [5] is said to be weakly sequentially continuous if $x_n \rightharpoonup x$ in E implies that $\{J(x_n)\}$ converges weak star to $J(x)$ in E^*. This happens, for example, if E is a Hilbert space, or finite-dimensional and smooth, or l^p, $1 < p < \infty$. This property of Banach spaces was introduced by Browder [1]. More information can be found in [6].

Lemma. *Let A be a maximal accretive set in a Banach space E whose norm is Gâteaux differentiable. Let $x_n \in D(A)$, $y_n \in Ax_n$, $x_n \rightharpoonup x$, and $y_n \to y$. If the duality mapping J is weakly sequentially continuous, then $[x, y] \in A$.*

Proof. Let $z \in D(A)$ and $w \in Az$. We have

$$\begin{aligned}
&|(y_n - w, J(x_n - z)) - (y - w, J(x - z))| \\
&\le |(y_n - y, J(x_n - z))| + |(y - w, J(x_n - z) - J(x - z))| \\
&\le \|y_n - y\|\,\|x_n - z\| + |(y - w, J(x_n - z) - J(x - z))|.
\end{aligned}$$

Thus

$$(y - w, J(x - z)) = \lim_{n\to\infty} (y_n - w, J(x_n - z)) \ge 0.$$

The result follows.

A closed subset C of a Banach space E is called a nonexpansive retract of E if there exists a retraction of E onto C which is a nonexpansive mapping. A retraction $P: E \to C$ is called a sunny retraction if $P(x) = v$ implies that $P(v + r(x - v)) = v$ for all $x \in E$ and $r \geq 0$. (We prefer this term to the one used by Bruck [3] because suns already occur in approximation theory.) If there exists a retraction $P: E \to C$ which is both sunny and nonexpansive, then C is said to be a sunny nonexpansive retract of E. If C is a sunny nonexpansive retract of a Banach space whose norm is Gâteaux differentiable, then the sunny nonexpansive retraction on C is unique [3, Theorem 1], [8, Lemma 2.7]. The metric projection on a closed and convex subset of a Hilbert space is both sunny and nonexpansive.

Theorem. *Let A be an m-accretive set in a reflexive Banach space E with a Gâteaux differentiable norm. If the duality mapping J of E is weakly sequentially continuous and $0 \in R(A)$, then for each x in E the strong $\lim_{r\to\infty} J_r x$ exists and belongs to $A^{-1}(0)$.*

Proof. Let the positive sequence $\{r_n : n = 1, 2, \dots\}$ tend to infinity. Let $x \in E$ and $y \in A^{-1}(0)$. Set $x_n = J_{r_n} x$. We have $(x_n - x, J(y - x_n)) \geq 0$ because $(x - x_n)/r_n$ belongs to Ax_n and 0 belongs to Ay. Consequently,

$$\|y - x_n\|^2 \leq (y - x, J(y - x_n)) \leq \|y - x\| \, \|y - x_n\|$$

and $\{x_n\}$ is bounded. Let Px be the weak limit of a subsequence $\{x_k\}$ of $\{x_n\}$. Clearly $(x - x_k)/r_k \to 0$. By the Lemma, $[Px, 0]$ belongs to A. Therefore

$$\|Px - x_k\|^2 \leq (Px - x, J(Px - x_k)) \to 0.$$

Thus $\{x_k\}$ converges strongly to Px. It follows that $(Px - x, J(y - Px)) \geq 0$ for all x in E and y in $A^{-1}(0)$. In other words [8, Lemma 2.7], $P: E \to A^{-1}(0)$ is both sunny and nonexpansive. Since P is necessarily unique, the sequence $\{x_n\}$ itself converges strongly to Px. This completes the proof.

Corollary (cf. [8, Theorem 3.2]). *Let T be a nonexpansive self-mapping of E, a reflexive Banach space with a Gâteaux differentiable norm. Suppose that T has a nonempty fixed point set and that E has a weakly sequentially continuous duality mapping. Let x belong to E. For each $0 < k < 1$ let x_k satisfy $x_k = kTx_k + (1 - k)x$. Then the strong $\lim_{k\to 1-} x_k$ exists and is a fixed point of T.*

In the course of the proof of the Theorem it has been established that $A^{-1}(0)$ is a nonexpansive retract of E. Since $A^{-1}(0)$ is the fixed point set of the nonexpansive mapping J_r (for all $r > 0$), this is also a consequence of [2, Theorem 2]. In a similar setting, $\mathrm{cl}(D(A))$ is also a nonexpansive retract of E [9, Theorem 3.7].

Remark. A version of the Theorem is true for a certain class of accretive operators which are not necessarily m-accretive.

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