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ON THE INFINITE DIMENSIONALITY OF THE DOLBEAULT COHOMOLOGY GROUPS

HENRY B. LAUFER[1]

ABSTRACT. Let M be an open subset of a Stein manifold without isolated points. Let Ω^p be the sheaf of germs of holomorphic p-forms on M. Then $H^q(M, \Omega^p)$ is either 0 or else infinite dimensional. $H^q(M, \mathcal{S})$ may be nonzero and finite dimensional if M is the regular points of a Stein space or if $\mathcal{S}$ is an arbitrary coherent sheaf over M.

Let M be a complex manifold. Let Ω^p be the sheaf of germs of holomorphic p-forms over M. If M is a Stein manifold, then $H^q(M, \Omega^p) = 0$ for $q \geq 1$ [3, VIII.A, 14, Cartan's Theorem B, p. 243] while $H^0(M, \Omega^p)$ is an infinite dimensional Fréchet space, so long as M does not consist of a finite number of points. If M is a compact manifold, then $H^q(M, \Omega^p)$ is finite dimensional for all p and all q [3, VIII.A, 10, p. 245]. In this paper, we shall examine the possible dimensions for $H^q(M, \Omega^p)$, the Dolbeault cohomology groups, under the assumption that M is an open subset of a Stein manifold and that M does not have any isolated points. By considering the natural topology on $H^q(M, \Omega^p)$ Siu showed [7, Theorem A, p. 17], under much more general assumptions, that $H^q(M, \Omega^p)$ cannot be countably infinite dimensional. The topology on $H^q(M, \Omega^p)$ is that induced from the topology of uniform convergence on compact sets for all derivatives for C^∞ differential forms. $H^q(M, \Omega^p)$ is then a linear topological space. Let $R^{p,q}$ be the closure of 0 in $H^q(M, \Omega^p)$. Then $H^q(M, \Omega^p)/R^{p,q}$ is a separable Fréchet space. Siu essentially showed that $R^{p,q}$ cannot be countably infinite dimensional. The main result of this paper is that $H^q(M, \Omega^p)/R^{p,q}$ is either 0 or infinite dimensional. The author previously proved a special case in [5, Theorem 4.5, p. 431]. Some examples are given which show that some special assumptions about M and about the sheaf are needed.

Let $E^{p,q}$ be the C^∞ differential forms on M of type (p, q). If f is a holomorphic function on M, then f operates on $E^{p,q}$ via multiplication. We shall also denote this endomorphism of $E^{p,q}$ by f. Let λ be a holomorphic vector field on M, i.e., a section of the dual sheaf to Ω^1. Then λ induces a map, also denoted by λ, $\lambda: E^{p,q} \to E^{p-1,q}$ given by contraction, thinking of

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the vector field λ and elements of $E^{p,q}$ and $E^{p-1,q}$ as tensors. In local coordinates, if

$$\lambda = \sum \lambda_k(z) \frac{\partial}{\partial z_k}, \qquad 1 \le k \le n,$$

and $\omega \in E^{p,q}$ is given by

$$\omega = \sum \omega_{I,J}(z)\, dz^I \wedge d\bar{z}^J,$$

where the summation is over the multi-indices $I = (i_1, \ldots, i_p)$, $i_1 < \cdots < i_p$, and $J = (j_1, \ldots, j_q)$, $j_1 < \cdots < j_q$, then

$$\lambda(\omega) = \sum (-1)^{\epsilon+1} \lambda_k(z) \omega_{I,J}(z)\, dz^{I'} \wedge d\bar{z}^J$$

where the summation is over $1 \le k \le n$, $J = (j_1, \ldots, j_q)$ and $I = (i_1, \ldots, i_p)$ such that $k \in I$. I' is obtained from I by deleting k. ϵ is given by $I \ni k$, $k = i_\epsilon$.

The map λ commutes with the map f.

Let $D: E^{p,q} \to E^{p,q}$ be given by $D = \partial \circ \lambda + \lambda \circ \partial$. The following lemma is true for any complex manifold. ∂f is a holomorphic 1-form and $\lambda(\partial f)$ is the function obtained by the usual operation of a vector field λ on a function f.

Lemma. *Let* $D = \partial \circ \lambda + \lambda \circ \partial$. *Then* $D \circ f - f \circ D = \lambda(\partial f)$.

Proof. Let $\omega \in E^{p,q}$.

(1) $$\begin{aligned}(\partial \circ \lambda \circ f)(\omega) = (\partial \circ f \circ \lambda)(\omega) &= \partial f \wedge \lambda(\omega) + f \wedge (\partial \circ \lambda)(\omega) \\ &= \partial f \wedge \lambda(\omega) + (f \circ \partial \circ \lambda)(\omega).\end{aligned}$$

(2) $$(\lambda \circ \partial \circ f)(\omega) = \lambda(\partial f \wedge \omega + f \wedge \partial\omega) = \lambda(\partial f \wedge \omega) + (f \circ \lambda \circ \partial)\omega.$$

From (1) and (2),

(3) $$(D \circ f - f \circ D)(\omega) = \partial f \wedge \lambda(\omega) + \lambda(\partial f \wedge \omega).$$

The right side of (3) is $\mathbb{C}$-linear in λ and C^∞-linear in ω. So to complete the verification of the Lemma, it suffices to evaluate the right side of (3) in local coordinates with $\lambda = \partial/\partial z_1$ and $\omega = dz^I \wedge d\bar{z}^J$. There are two cases, $1 \notin I$ and $1 = i_1 \in I = (i_1, \ldots, i_p)$.

For $1 \notin I$: $\lambda(\omega) = 0$ and $\lambda(\partial f \wedge \omega) = \partial f/\partial z_1 \cdot \omega$, as needed.

For $1 \in I$: $\lambda(\omega) = dz^{I'} \wedge d\bar{z}^J$ where $I' = (i_2, \ldots, i_p)$.

(4) $$\partial f \wedge \lambda(\omega) = \sum \frac{\partial f}{\partial z_k} dz^k \wedge dz^{I'} \wedge d\bar{z}^J, \qquad k \notin I',$$

$$\partial f \wedge \omega = \sum \frac{\partial f}{\partial z_k} dz^k \wedge dz^I \wedge d\bar{z}^J, \qquad k \notin I.$$

(5) $$\lambda(\partial f \wedge \omega) = \sum (-1) \frac{\partial f}{\partial z_k} dz^k \wedge dz^{I'} \wedge d\bar{z}^J, \qquad k \notin I.$$

$I - I' = \{1\}$ so the summations in (4) and (5) differ only by one term, $k = 1$. This completes the proof of the Lemma.

Theorem. *Let M be an open subset of a Stein manifold having no isolated points. Then, for any p and q, $H^q(M, \Omega^p)$ is either 0 or else infinite dimensional. Let $R^{p,q}$ be the closure of 0 in $H^q(M, \Omega^p)$. Then $H^q(M, \Omega^p)/R^{p,q}$ is either 0 or else infinite dimensional.*

Proof. The proof for $H^q(M, \Omega^p)$ is exactly like that for $H^q(M, \Omega^p)/R^{p,q}$, leaving out topological considerations; therefore we omit it.

For the sake of notational simplicity, let $H = H^q(M, \Omega^p)/R^{p,q}$. We shall show that if H is finite dimensional, then $H = 0$. For f holomorphic on M, the action of f on $E^{p,q}$ is continuous and commutes with $\bar{\partial}$. Thus, f induces a continuous endomorphism on $H^q(M, \Omega^p)$ and on H.

Without loss of generality, we may assume that M is a subset of a connected Stein manifold S of dimension $n > 0$. Let I be the set of holomorphic functions on S which, after restriction to M, induce the zero map of H to itself. It suffices to show that $1 \in I$. Let $z_1, \dots, z_{2n+1}$ be holomorphic functions on S which separate points [3, Theorem VII.C.10, p. 224]. Each z_i acts on H and so has a minimal polynomial $p_i(z_i)$, under the finite dimensional assumption on H. $p_i(z_i) \in I$. $p_i(z_i)$ has only a finite number of roots in z_i. So the common zeros on S for $p_1(z_1), \dots, p_{2n+1}(z_{2n+1})$ consist of only a finite number of points, say $P_1, \dots, P_N$, in S.

∂: $E^{p,q} \to E^{p+1,q}$ is continuous and anticommutes with $\bar{\partial}$. If λ is a holomorphic vector field on S, then λ: $E^{p,q} \to E^{p-1,q}$ is continuous and commutes with $\bar{\partial}$. So $\lambda \circ \partial$ and $\partial \circ \lambda$ both induce endomorphisms of H. The Lemma also holds for the induced maps on H. Thus, if $f \in I$, then also $\lambda(\partial f) \in I$. Consider a $P_j \in S$ from above. Let $f \in I$ have a zero at P_j of minimal total order. $f \not\equiv 0$ since $p_1(z_1) \in I$. We claim that $f(P_j) \neq 0$, for suppose otherwise. Since S is a Stein manifold of positive dimension, by an application of Cartan's Theorem B, we can specify the tangent vector at P_j for a vector field λ on S. For a suitable choice for λ, $\lambda(\partial f)$ will have a zero of lower total order at P_j than has f. So for each P_j, there exists an $f_j \in I$ such that $f_j(P_j) \neq 0$. $p_1(z_1), \dots, p_{2n+1}(z_{2n+1}), f_1, \dots, f_N$ are then elements of I with no common zeroes. By [3, Corollary VIII.A.16, p. 244] there exist holomorphic functions $\{g_k\}$ on S such that $\Sigma g_i p_i(z_i) + \Sigma g_j f_j \equiv 1$. Thus $1 \in I$ and $H = 0$, as desired.

The Theorem does not hold under the weaker assumption that M is an open subset of a Stein *space*, even if M itself is a manifold. Consider, for example, a Riemann surface R of genus at least 1 embedded as the 0-section of a negative vector bundle V of rank 4. See [2]. V can be taken to be the direct sum of 4 line bundles, each of negative Chern class. Let $\mathcal{O} = \Omega^0$ be the sheaf of germs of holomorphic functions. Then $H^1(V, \mathcal{O}) \neq 0$ since $H^1(V, \mathcal{O})$ may be expanded in a power series along the fibers [2, pp. 343–344]. Also, $H^1(V, \mathcal{O})$ is finite dimensional since V is strictly pseudoconvex [1, Theorem 11, p. 239]. Let $M = V - R$. Then the restriction map induces an isomorphism $H^1(V, \mathcal{O}) \approx H^1(M, \mathcal{O})$ by [6, Corollary, p. 351]. Thus $H^1(M, \mathcal{O})$ is nonzero and finite dimensional. By blowing down R to a point p, we obtain a Stein space S with M the complement of the singular point p.

This example also shows that Ω^p in the Theorem cannot be replaced by an arbitrary coherent sheaf. Namely, near p, S may be embedded as a subvariety X of a polydisc Δ.

$H^1(S - p, \mathcal{O}) \approx H^1(X - p, \mathcal{O})$ by [4, Theorem 2.2, p. 105]. ${}_X\mathcal{O}$ is a coherent sheaf on Δ, but $H^1(\Delta - p, {}_X\mathcal{O})$ is nonzero and finite dimensional.

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DEPARTMENT OF MATHEMATICS, STATE UNIVERSITY OF NEW YORK, STONY BROOK, NEW YORK 11794