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DISK-LIKE PRODUCTS OF λ CONNECTED CONTINUA. II

CHARLES L. HAGOPIAN

ABSTRACT. R. H. Bing [3] proved that every atriodic, hereditarily decomposable, hereditarily unicoherent continuum is arc-like. Using this theorem, the author [5] showed that λ connected continua X and Y are arc-like when the topological product $X \times Y$ is disk-like. In this paper we consider products that have a more general mapping property. Suppose that X and Y are λ connected continua and that for each $\epsilon > 0$, there exists an ϵ-map of $X \times Y$ into the plane. Then X is either arc-like or circle-like. Furthermore, if X is circle-like, then Y is arc-like. Hence $X \times Y$ is either disk-like or annulus-like.

Throughout this paper a *continuum* is a nondegenerate, compact, connected metric space. A continuous function f of a continuum X is called an *ϵ-map* if for each point y of $f(X)$, the diameter of $f^{-1}(y)$ is less than ϵ.

A continuum X is *arc-like* if for each positive number ϵ, there exists an ϵ-map of X onto an arc. *Circle-like, disk-like,* and *annulus-like* continua are defined in the same manner. Here a disk is a 2-cell and an annulus is a planar continuum that is homeomorphic to the product obtained by crossing an arc with a circle.

A *chain* is a finite sequence $L_1, L_2, \ldots, L_n$ of open sets such that $L_i \cap L_j \neq \emptyset$ if and only if $|i-j| \leq 1$. If L_1 also intersects L_n, the sequence is called a *circular chain.* Each L_i is called a *link.* A chain is called an *ϵ-chain* if the diameter of each of its links is less than ϵ. An *ϵ-circular chain* is defined similarly. It follows from Urysohn's lemma that a continuum X is arc-like (circle-like) if and only if for each $\epsilon > 0$, X can be covered by an ϵ-chain (ϵ-circular chain).

A continuum X is said to have *property* A at a point x of X if every subcontinuum L of X that contains x is irreducible between x and some other point of L.

A continuum is *decomposable* if it is the union of two proper subcontinua. A continuum is *hereditarily decomposable* if all of its subcontinua are decomposable. If every pair of points in a continuum X lies in a hered-

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itarily decomposable subcontinuum of X, then X is said to be λ *connected.*

A continuum T is called a *triod* if it contains a subcontinuum Z such that $T - Z$ is the union of three nonempty disjoint open sets. When a continuum does not contain a triod, it is said to be *atriodic.*

A continuum is *unicoherent* provided that if it is the union of two subcontinua E and F, then $E \cap F$ is connected. A continuum is *hereditarily unicoherent* if all of its subcontinua are unicoherent.

For any two metric spaces (X, ψ) and (Y, ϕ), we shall always assume that the distance between two points $p_1 = (x_1, y_1)$ and $p_2 = (x_2, y_2)$ of the topological product $X \times Y$ is defined by

$$\rho(p_1, p_2) = ((\psi(x_1, x_2))^2 + (\phi(y_1, y_2))^2)^{1/2}.$$

We shall denote the closure and the boundary of a given set Z by Cl Z and Bd Z respectively.

Lemma 1. *Suppose that X and Y are continua, that X is not unicoherent, and that v_1, v_2, and v_3 are distinct points of Y and R is a continuum such that $\{v_1, v_2\} \subset R \subset Y - \{v_3\}$. There exists an $\epsilon > 0$ such that if g is an ϵ-map of $X \times Y$ into the plane, then some element of $\{g(X \times \{v_1\}), g(X \times \{v_2\}), g(X \times \{v_3\})\}$ separates the plane between the other two.*

Proof. Since X is not unicoherent, there exist continua E and F and nonempty disjoint closed sets A and B such that $X = E \cup F$ and $A \cup B = E \cap F$. Define C_1 and C_2 to be open subsets of X such that $A \subset C_1$, $B \subset C_2$, $\mathrm{Cl}\ C_1 \cap \mathrm{Cl}\ C_2 = \emptyset$.

Let ψ and ϕ be distance functions for X and Y respectively. Define ϵ to be a positive number less than $\psi(C_1, C_2)$, $\psi(E, F - (C_1 \cup C_2))$, $\psi(F, E - (C_1 \cup C_2))$, $\phi(v_1, v_2)$, and $\phi(v_3, R)$. It follows directly from paragraphs 7 through 12 in the proof of Theorem 1 in [5] that if g is an ϵ-map of $X \times Y$ into the plane, then g has the required condition.

Theorem 1. *Suppose that X and Y are continua and that for each $\epsilon > 0$, there exists an ϵ-map of $X \times Y$ into the plane. Then X is atriodic, every proper subcontinuum of X is unicoherent, and Y is unicoherent when X is not unicoherent.*

Proof. It follows from paragraphs 2 through 4 in the proof of Theorem 1 in [5] that X is atriodic. By the argument presented in paragraphs 5 through 13 in the same proof, every proper subcontinuum of X is unicoherent.

Suppose that neither X nor Y is unicoherent. There exist continua R and T and nonempty disjoint closed sets C and D such that $Y = R \cup T$ and $C \cup D = R \cap T$. Let c and d be points of C and D respectively. There exist points v and w of $Y - (C \cup D)$ and continua V and W such that $\{d, v\} \subset V \subset R - \{c\}$ and $\{d, w\} \subset W \subset T - \{c\}$.

Note that for any three points of $\{v, w, c, d\}$, some element of $\{R, T\}$ contains two and misses the other. According to Lemma 1, there exists a positive number ϵ such that if g is an ϵ-map of $X \times Y$ into the plane E^2, then for any three elements of $\{g(X \times \{v\}), g(X \times \{w\}), g(X \times \{c\}), g(X \times \{d\})\}$, one must separate E^2 between the other two.

Letting ϕ denote the metric on Y, we define ϵ' to be a positive number less than ϵ, $\phi(v, T)$, $\phi(w, R)$, and $\phi(c, V \cup W)$. Let h be an ϵ'-map of $X \times Y$ into E^2. Since $h(X \times R) \cap h(X \times \{w\}) = \varnothing$ and $h(X \times \{c\}) \cap h(X \times W) = \varnothing$, $h(X \times \{d\})$ separates $h(X \times \{c\})$ from $h(X \times \{w\})$ in E^2. Furthermore, since $h(X \times T) \cap h(X \times \{v\}) = \varnothing$ and $h(X \times \{c\}) \cap h(X \times V) = \varnothing$, $h(X \times \{d\})$ separates $h(X \times \{c\})$ from $h(X \times \{v\})$ in E^2.

Define H to be the union of $h(X \times \{d\})$ with all components of $E^2 - h(X \times \{d\})$ that meet $h(X \times \{v, w\})$. Note that H is a connected set that contains $h(X \times \{v, w\})$ and misses $h(X \times \{c\})$. Thus $h(X \times \{c\})$ does not separate $h(X \times \{v\})$ from $h(X \times \{w\})$ in E^2. It follows that one element of $\{h(X \times \{v\}), h(X \times \{w\})\}$ separates the other from $h(X \times \{c\})$. We suppose without loss of generality that $h(X \times \{v\})$ separates $h(X \times \{w\})$ from $h(X \times \{c\})$. Since $h(X \times T)$ is a continuum in $E^2 - h(X \times \{v\})$ that contains $h(X \times \{c\})$ and $h(X \times \{w\})$, we have a contradiction. Hence Y is unicoherent when X is not unicoherent.

Lemma 2. *If X is a continuum that is not unicoherent and if every proper subcontinuum of X is unicoherent, then X is not separated by any of its subcontinua.*

Proof. Assume there is a continuum H in X such that $X - H$ is not connected. Since X is not unicoherent, there exist continua E and F and nonempty disjoint closed sets A and B such that $X = E \cup F$ and $A \cup B = E \cap F$.

The set $F - E$ is connected; for otherwise, the closure of some component K of $F - E$ would meet both A and B [7, Theorem 50, p. 18] and $E \cup K$ would be a proper subcontinuum of X that is not unicoherent. Note that both E and $F' = \mathrm{Cl}(F - E)$ are irreducible between $A' = F' \cap A$ and $B' = F' \cap B$. Hence H intersects $A' \cup B'$.

Assume that $H \cap A' \neq \varnothing$ and that $H \cap B' = \varnothing$. Since $H \cup E$ and $H \cup F'$ are proper subcontinua of X, the sets $H \cap E$ and $H \cap F'$ are connected. Let q be a point of B' and define Y to be the continuum $(H \cap E) \cup (q$-component of $E - H)$. Define Z to be the continuum $(H \cap F') \cup (q$-component of $F' - H)$. It follows that $Y \cup Z$ is a proper subcontinuum of X and $Y \cap Z$ is not connected, which is impossible. Thus if $H \cap A' \neq \varnothing$, then $H \cap B' \neq \varnothing$.

By the same argument, $H \cap A' \neq \varnothing$ when $H \cap B' \neq \varnothing$. Hence H meets both A' and B'. It follows that $H \supset E$ or $H \supset F'$.

Assume without loss of generality that E lies in H. Since F' is irreducible between A' and B', it follows that $H \cap F'$ is not connected. Let P and Q be nonempty closed disjoint sets such that $H \cap F' = P \cup Q$. There exists a component N of $F' - H$ such that Cl N meets both P and Q. Since $X - H$ is not connected, $H \cup N$ is a proper subcontinuum of X. But since $H \cap \text{Cl } N$ is not connected, we have a contradiction. It follows that X is not separated by any of its subcontinua.

Theorem 2. *If X is an atriodic λ connected continuum that is not unicoherent and if every proper subcontinuum of X is unicoherent, then X is circle-like.*

Proof. We begin by showing that X is hereditarily decomposable. Then using a decomposition theorem of Vought's and results of Bing's involving arc-like continua, we construct an ϵ-circular chain that covers X.

Assume that X contains an indecomposable continuum I. Since X is λ connected, I is a proper subcontinuum of X [7, Theorem 139, p. 59]. Let A and B be distinct composants of I. Let H be a hereditarily decomposable subcontinuum of X that meets A and B. Since H does not contain I, it follows that $H \cap I$ is not connected and $H \cup I = X$.

Suppose that $H \cap I$ is the union of three nonempty disjoint closed sets C, D, and E. Define C', D', and E' to be disjoint open subsets of X containing C, D, and E respectively. It follows from [7, Theorem 50, p. 18] that $I \cup C' \cup D' \cup E'$ contains a triod; which contradicts our hypothesis. Hence $H \cap I$ is the union of two closed connected sets in $A \cup B$.

Since $H \cup I = X$, it follows that A and B are the only composants of I that can be joined by a hereditarily decomposable subcontinuum of X; which contradicts the assumption that X is λ connected. Hence X is hereditarily decomposable.

It follows from Lemma 2 and [8, Theorem 2] that X has a monotone upper semicontinuous decomposition $\mathcal{G}$ each of whose elements has void interior and whose quotient space is a circle S^1. Let f denote the quotient map of X onto S^1 associated with $\mathcal{G}$. If G is a nondegenerate element of $\mathcal{G}$, then G is an atriodic, hereditarily decomposable, hereditarily unicoherent continuum, and therefore has property A at one of its points [3, Theorems 8 and 9].

Define p_1 and p_2 to be distinct points of S^1. Let U_1 and U_2 be the components of $S^1 - \{p_1, p_2\}$. For $i = 1$ and 2, let $\mathcal{G}_i$ be the collection consisting of all elements of $\mathcal{G}$ that lie in $f^{-1}(\text{Cl } U_i)$.

Note that for $i = 1$ and 2, there exists an uncountable subcollection $\mathcal{F}_i$ of $\mathcal{G}_i$ such that for each element F of $\mathcal{F}_i$, there is a sequence of elements of $\mathcal{G}_i$ converging to F such that infinitely many elements of this sequence separate F from $f^{-1}(p_1)$ and infinitely many separate F from

$f^{-1}(p_2)$ in $f^{-1}(\mathrm{Cl}\ U_i)$. For $i = 1$ and 2, let F_i be an element of $\mathfrak{F}_i$. For $i = 1$ and 2, if F_i is not degenerate, define x_i to be a point at which F_i has property A; otherwise, define x_i to be the only point in F_i.

Now let ϵ be a positive number. Define V_1 and V_2 to be the components of $S^1 - \{f(F_1), f(F_2)\}$. For $i = 1$ and 2, the atriodic, hereditarily decomposable, hereditarily unicoherent continuum $f^{-1}(\mathrm{Cl}\ V_i)$ is arc-like [3, Theorem 11]. Moreover, each $f^{-1}(\mathrm{Cl}\ V_i)$ has property A at x_1 and x_2 and is irreducible between these points. Hence there exist ϵ-chains $H_1, H_2, \ldots, H_m$ and $K_1, K_2, \ldots, K_n$ covering $f^{-1}(\mathrm{Cl}\ V_1)$ and $f^{-1}(\mathrm{Cl}\ V_2)$, respectively, such that x_1 is a point of $(H_1 - \mathrm{Cl}\ H_2) \cap (K_n - \mathrm{Cl}\ K_{n-1})$, x_2 belongs to $(H_m - \mathrm{Cl}\ H_{m-1}) \cap (K_1 - \mathrm{Cl}\ K_2)$, and some element of $H_1, H_2, \ldots, H_m$ misses every element of $K_1, K_2, \ldots, K_n$ [3, Theorems 13 and 14], [6, Lemma (proof)].

Let M be an open subset of $(H_1 - \mathrm{Cl}\ H_2) \cap (K_n - \mathrm{Cl}\ K_{n-1})$ that contains x_1. Let N be an open subset of $(H_m - \mathrm{Cl}\ H_{m-1}) \cap (K_1 - \mathrm{Cl}\ K_2)$ that contains x_2. Then $X - (M \cup N)$ is the union of nonempty closed disjoint sets A and B such that B contains $f^{-1}(\mathrm{Cl}\ V_2) - (M \cup N)$ and is covered by $K_1, K_2, \ldots, K_n$. An ϵ-circular chain covering X is $(A \cap H_1) \cup M$, $A \cap H_2, \ldots, A \cap H_{m-1}, (A \cap H_m) \cup N, (B \cap K_1) \cup N, B \cap K_2, \ldots, (B \cap K_n) \cup M$.

Theorem 3. *Suppose that X and Y are λ connected continua and that for each $\epsilon > 0$, there exists an ϵ-map of $X \times Y$ into the plane. Then X is either arc-like or circle-like. If X is circle-like, then Y is arc-like.*

Proof. According to Theorem 1, X is atriodic and every proper subcontinuum of X is unicoherent.

First assume that X is unicoherent. It follows that X is hereditarily decomposable [5, Theorem 2]. Hence X is arc-like [3, Theorem 11]. It follows from [4, Theorem 7] that X is not circle-like.

Now suppose that X is not unicoherent. By Theorem 2, X is circle-like. Note that Y is atriodic, hereditarily unicoherent (Theorem 1), and hereditarily decomposable [5, Theorem 2]. Thus Y is arc-like.

Theorem 4. *If X and Y are λ connected continua and if for each $\epsilon > 0$, there exists an ϵ-map of $X \times Y$ into the plane, then $X \times Y$ is either disk-like or annulus-like.*

Proof. If f is an $\epsilon/2$-map of X onto a space A and if g is an $\epsilon/2$-map of Y onto a space B, then the function h of $X \times Y$ onto $A \times B$ defined by $h((x, y)) = (f(x), g(y))$ is an ϵ-map. Hence this theorem follows from Theorem 3.

The cone over the dyadic solenoid is a disk-like continuum that cannot be embedded in E^3 (Euclidean 3-space) [2].

Question. If a product of two continua is disk-like, must it be embeddable in E^3?

Note that since the product of any two arc-like continua is embeddable in E^3 [1], the answer to this question is "yes" if the answer to Question 1 of [5] is "yes".

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DEPARTMENT OF MATHEMATICS, CALIFORNIA STATE UNIVERSITY, SACRAMENTO, CALIFORNIA 95819