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THE DERIVATIVE OF A BOUNDED HOLOMORPHIC FUNCTION IN THE DISK

SHINJI YAMASHITA

ABSTRACT. Let a nonconstant function f be holomorphic and bounded, $|f| < 1$ in $D: |z| < 1$. We shall estimate $f^*(z) = (1 - |z|^2)|f'(z)|/(1 - |f(z)|^2)$ at each point $z \in D$ ((1) in Theorem 1). The function d appearing in the estimate concerns the sizes of the schlicht disks on the Riemannian image $\mathfrak{F}$ of D by f. Boundary properties of f and f^* will be stated in Theorems 2 and 3; use is made of the cluster sets of d.

1. Results. Consider the metric

$$\delta(w, z) = |w - z|/|1 - \bar{z}w| \qquad (z, w \in D)$$

in D ($\delta(w, z) = [w, z]$ in [8, p. 510 ff.]). Denote $D(z, t) = \{w \in D; \delta(w, z) < t\}$ ($z \in D$, $0 < t \leq 1$). *By f we always mean a nonconstant function holomorphic and bounded,* $|f| < 1$ *in* D. Let $d(z) \equiv d(z, f)$ be the maximum of t such that $\mathfrak{F}$, covering D, contains the schlicht $D(f(z), t)$ of center $f(z) \in \mathfrak{F}$; if such a t does not exist we set $d(z) = d(z, f) = 0$. We then have $0 \leq d(z) \leq 1$ at each $z \in D$ and further, $d(z) = 1$ at (each) $z \in D$ if and only if f is schlicht from D onto D. Plainly, $d(z) = 0$ if and only if $f'(z) = 0$.

Theorem 1. *At each $z \in D$ we have*

$$d(z, f) \leq f^*(z) \leq \{8d(z, f)\}^{1/2}. \tag{1}$$

Since $f^*(z) \leq 1$ at each $z \in D$ by the lemma of H. A. Schwarz and G. Pick, the right-hand side of (1) has the meaning if $d(z, f) < 1/8$. Theorem 1 is analogous to the result of Ch. Pommerenke [7, Theorem 1] who uses the Euclidean distance to measure the sizes of the schlicht disks on $\mathfrak{F}$.

By an angular domain at a point ζ of $\Gamma: |z| = 1$, we mean a triangular domain whose vertices are ζ and two points of D. By an admissible arc at ζ we mean a continuous curve $\Lambda: z = z(t) \in D$ $(0 \leq t < 1)$, $\lim_{t \to 1} z(t) = \zeta$, tangent at ζ to a chord of Γ at ζ. We call $\zeta \in \Gamma$ of first kind if

$$\liminf_{z \to \zeta, z \in \Lambda} f^*(z) = 0$$

for each admissible arc Λ at ζ, while we call $\zeta \in \Gamma$ of second kind if $\liminf_{z \to \zeta, z \in \Delta} f^*(z) > 0$ for each angular domain Δ at ζ. A point $\zeta \in \Gamma$ is

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called of third kind if $\liminf_{z\to\zeta, z\in D} f^*(z) > 0$. Let $S_1(f)$, $S_2(f)$ and $S_3(f)$ be the sets of all points of first, second, and third kind, respectively. Plainly, $S_3(f) \subset S_2(f)$ and $S_1(f) \cap S_2(f) = \emptyset$ (empty). Our next results are two.

Theorem 2. *The union* $S_1(f) \cup S_2(f)$ *is of linear measure* 2π *and residual* [2, *p.* 75] *on* Γ, *and the union* $S_1(f) \cup S_3(f)$ *is residual on* Γ.

Let

$$\sigma(w, z) = \tfrac{1}{2} \log\{(1 + \delta(w, z))/(1 - \delta(w, z))\}$$

be the non-Euclidean distance of w and z of D. Let $l(w, z; f)$ $(w, z \in D)$ be the non-Euclidean length of the Riemannian image by f of the (Euclidean) line segment joining w and z.

Theorem 3. *If* Δ *is an angular domain at* $\zeta \in S_2(f)$, *then there exist an open disk* U *containing* ζ *and a constant* $k_1 > 0$ *such that* $k_1\sigma(w, z) \le l(w, z; f)$ *for each pair of points* $w, z \in \Delta \cap U$. *If* $\zeta \in S_3(f)$, *then there exist an open disk* V *containing* ζ *and a constant* $k_2 > 0$ *such that* $k_2\sigma(w, z) \le l(w, z; f)$ *for each pair of points* $w, z \in D \cap V$.

2. **Proof of Theorem 1.** We have only to prove (1) for z with $d(z) > 0$. The function

$$g(w) = \frac{f((w + z)/(1 + \bar{z}w)) - f(z)}{1 - \overline{f(z)}\, f((w + z)/(1 + \bar{z}w))}$$

is holomorphic and bounded, $|g| < 1$ in $|w| < 1$ with $g(0) = 0$ and $|g'(0)| = f^*(z) \neq 0$. Therefore the function $h(w) \equiv g(w)/g'(0) = w + a_2 w^2 + \cdots$ is bounded by $M = 1/|g'(0)|$ in $|w| < 1$. To the function h we may apply the result [8, Corollary, p. 261] deduced from the theorem [8, Theorem VI.10, p. 259] due to J. Dieudonné [3, p. 349 ff.]. Then the Riemannian image of $|w| < 1$ by h contains the schlicht disk $D(0, 1/8M) = D(0, |g'(0)|/8)$, whence the Riemannian image of $|w| < 1$ by g contains the schlicht disk $D(0, |g'(0)|^2/8)$. Therefore $\mathfrak{F}$ contains the schlicht disk $D(f(z), |g'(0)|^2/8)$, whence $f^*(z)^2/8 = |g'(0)|^2/8 \le d(z)$. This proves the right-hand side of (1). Let F be the inverse of f in $D(f(z), d(z))$ such that $F(f(z)) = z$. The function

$$G(w) = \frac{F((d(z)w + f(z))/(1 + \overline{f(z)}d(z)w)) - z}{1 - \bar{z}F((d(z)w + f(z))/(1 + \overline{f(z)}d(z)w))}$$

is holomorphic and bounded, $|G| < 1$ in $|w| < 1$ with $G(0) = 0$. Consequently, by Schwarz' lemma, $d(z)/f^*(z) = |G'(0)| \le 1$, from which follows the left-hand side of (1).

3. **Proofs of Theorems 2 and 3.** We begin with

Lemma 1. *For each pair* $w, z \in D$,

$$|d(w) - d(z)| \leq \sigma(w, z). \tag{2}$$

Proof. (The present proof, due to the referee, is more concise than the original one.) Without loss of generality we may assume that $d(z) < d(w)$. Then, $d(w) < 1$, for otherwise, $d(z) = d(w) = 1$. We may further assume $\sigma(w,z) < d(w)$, for otherwise, (2) is trivial. Since $\delta(f(w), f(z)) \leq \delta(w, z) \leq \sigma(w, z) < d(w)$, $f(z)$ lies in the disk $D(f(w), d(w))$. Hence the two disks $D(f(w), d(w))$ and $D(f(z), d(z))$ must possess a common boundary point $v \in D$, so that $\delta(f(w), v) = d(w)$ and $\delta(f(z), v) = d(z)$. It follows from $d(z) < d(w)$ that $\sigma(f(z), v) < \sigma(f(w), v)$. Now, making use of the fact that $\delta(f(w), v) = \tanh \sigma(f(w), v)$, $\delta(f(z), v) = \tanh \sigma(f(z), v)$, the fact that $\tanh A - \tanh B \leq A - B$ for $A \geq B \geq 0$, the triangle inequality for the metric σ, and the Schwarz-Pick lemma, we obtain the following chain of inequalities:

$$\begin{aligned} d(w) - d(z) &= \delta(f(w), v) - \delta(f(z), v) \leq \sigma(f(w), v) - \sigma(f(z), v) \\ &\leq \sigma(f(w), f(z)) \leq \sigma(w, z). \end{aligned}$$

This completes the proof of the lemma.

Let E be a subset of D whose closure $\overline{E}$ in the plane contains a point $\zeta \in \Gamma$. Then we set $C_E(d, \zeta) = \bigcap_U \overline{d(E \cap U)}$, where U ranges over all open disks containing ζ. The cluster set $C_E(d, \zeta)$ relative to E lies in the closed interval $[0, 1]$. We set $C^A(d, \zeta) = \bigcup_\Delta C_\Delta(d, \zeta)$ and $\Pi_T(d, \zeta) = \bigcap_\Lambda C_\Lambda(d, \zeta)$, Δ ranging over all angular domains at ζ and Λ ranging over all admissible arcs at ζ. Let $K(d)$ be the set of all points $\zeta \in \Gamma$ such that $C^A(d, \zeta) = C_\Delta(d, \zeta)$ for each angular domain Δ at ζ and let $J(d)$ be the set of all points $\zeta \in K(d)$ satisfying $C^A(d, \zeta) = C_D(d, \zeta)$. Finally, let $L(d)$ be the set of all points $\zeta \in \Gamma$ such that $C^A(d, \zeta) = \Pi_T(d, \zeta)$.

Lemma 2. $J(d) \subset K(d) = L(d)$.

Proof. $J(d) \subset K(d)$ follows from the definition of $J(d)$. Since each angular domain at $\zeta \in L(d)$ contains a terminal part of a chord of Γ at ζ, it follows from $C^A(d, \zeta) = \Pi_T(d, \zeta)$ that $C^A(d, \zeta) = C_\Delta(d, \zeta)$ for each angular domain Δ at ζ, that is, $\zeta \in K(d)$. Thus $L(d) \subset K(d)$. To prove $L(d) \supset K(d)$, we remark that d is uniformly continuous as a map from D endowed with the metric $\sigma(\cdot, \cdot)$ into $[0, 1]$. The proof is therefore the same as that of [1, Lemma].

Proof of Theorem 2. First of all, we may replace $f^*(z)$ by $d(z, f)$ in the definition of $S_j(f)$, $j = 1, 2, 3$; this is a consequence of (1). We shall prove

$$K(d) \subset S_1(f) \cup S_2(f); \tag{3}$$

$$J(d) \subset S_1(f) \cup S_3(f). \tag{4}$$

By Lemma 2, $K(d) = L(d)$. If $\zeta \in K(d)$ and $0 \in C^A(d, \zeta)$, then $0 \in \Pi_T(d, \zeta)$, whence $\zeta \in S_1(f)$, while if $\zeta \in K(d)$ and $0 \notin C^A(d, \zeta)$, then $0 \notin C_\Delta(d, \zeta)$ for each Δ at ζ, whence $\zeta \in S_2(f)$. We thus obtain (3). If $\zeta \in J(d)$ and $0 \in C_D(d, \zeta)$, then $0 \in \Pi_T(d, \zeta) = C^A(d, \zeta) = C_D(d, \zeta)$ because $J(d) \subset L(d)$. Therefore $\zeta \in S_1(f)$. If $\zeta \in J(d)$ and $0 \notin C_D(d, \zeta)$, then $\zeta \in S_3(f)$. We thus have (4). Now, according to the results on arbitrary functions (cf. [6, Theorems 1 and 2]), $K(d)$ is of measure 2π and $J(d)$ is residual on Γ. Hence $K(d) \supset J(d)$ is residual. Theorem 2 now follows from (3) and (4).

Remark. Let a real function q in D be uniformly continuous with respect to σ. Then $J(q) \subset K(q) = L(q)$ by the identical reasoning as in the proof of Lemma 2. Assume further that $q(z) \geq 0$ for all $z \in D$. If q replaces f^* in the definitions of $S_j(f)$ $(j = 1, 2, 3)$, and if the resulting sets are denoted by $S_j(q)$ $(j = 1, 2, 3)$, then Theorem 2 for the present q remains valid by the same proof. Theorem 1 reveals the link between f^* and d, the special case of q.

Proof of Theorem 3. Let U be an open disk containing ζ such that $k_1 = \inf_{z \in \Delta \cap U} f^*(z) > 0$. It then follows that

$$k_1 |d\xi|/(1 - |\xi|^2) \leq |f'(\xi)|\, |d\xi|/(1 - |f(\xi)|^2) \tag{5}$$

for each $\xi \in \Delta \cap U$. Since $k_1 \sigma(w, z)$ is obtained by integrating the left-hand side of (5) along the geodesic line connecting w and z, $k_1 \sigma(w, z)$ is less than the integral of the left-hand side along the Euclidean line segment joining z and w, being contained in the convex set $\Delta \cap U$. We thus obtain $k_1 \sigma(w, z) \leq l(w, z; f)$. The proof of the rest is similar.

4. A special class of functions. We consider the distribution of $S_1(f)$ of a special f. Suppose

$$\iint_{|z|<1} \left(\frac{|f'(z)|}{1 - |f(z)|^2} \right)^p dx\, dy < +\infty \qquad (1 \leq p < +\infty,\ z = x + iy). \tag{6}$$

Consider the case $p = 1$. By G. Fubini's theorem with $\frac{1}{2} \leq r$, we obtain

$$\int_{1/2}^{1} \frac{|f'(r\zeta)|}{1 - |f(r\zeta)|^2}\, dr < +\infty \tag{7}$$

for a.e. (almost every) ζ on Γ. Now (7) means that the non-Euclidean length of the Riemannian image of half the radius at ζ by f is finite. Hence, f has the radial limit $f(\zeta) \in D$ at ζ, which is the angular limit by the theorem of E. Lindelöf [2, Theorem 2.3, p. 19]. By a geometrical consideration on d combined with (1) we have

$$\lim_{z \to \zeta,\, z \in \Delta} d(z, f) = \lim_{z \to \zeta,\, z \in \Delta} f^*(z) = 0 \tag{8}$$

for each angular domain Δ at ζ; in effect, $d(z, f) \leq \delta(f(z), f(\zeta))$ for $z \in \Delta$

near ζ. Thus, $S_1(f)$ is of measure 2π. The case $p > 1$. The function

$$|f'(z)|/(1 - |f(z)|^2) = \exp \circ \log\{|f'(z)|/(1 - |f(z)|^2)\}$$

is nonnegative subharmonic in D; actually, the exponential function is convex and even $\log\{|f'(z)|/(1 - |f(z)|^2)\}$ is subharmonic because

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\log\left(\frac{|f'(z)|}{1 - |f(z)|^2}\right) = 4\left(\frac{|f'(z)|}{1 - |f(z)|^2}\right)^2 > 0$$

at each $z = x + iy$ with $f'(z) \neq 0$ (cf. [5, p. 83]). By the result of F. W. Gehring [4, Theorem 1], for a.e. $\zeta \in \Gamma$ we have

$$\lim_{z \to \zeta, z \in \Delta} \frac{(1 - |z|)^{1/p}|f'(z)|}{1 - |f(z)|^2} = 0$$

for each angular domain Δ at ζ. Consequently,

$$\lim_{z \to \zeta, z \in \Delta} (1 - |z|)^{(1/p) - 1} f^*(z) = 0 \tag{9}$$

for each angular domain Δ at ζ. Hence f^* tends to zero rapidly as (9) shows in this case. It should be noted that for $p = 2$ (hence for $p \geq 2$), we have (7) and, hence, (8) for each $\zeta \in \Gamma$ except for a set of capacity zero on Γ (cf. [9, Theorem for $j = 3$]). Therefore $\Gamma - S_1(f)$ is of capacity zero.

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DEPARTMENT OF MATHEMATICS, TOKYO METROPOLITAN UNIVERSITY, SETAGAYA-KU, TOKYO, 158, JAPAN