PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 53, Number 1, November 1975

REMARKS ON DILATIONS IN L_p-SPACES

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ABSTRACT. Let $(X, \mathfrak{F}, \mu)$ be a nonatomic measure space. It is shown that there exists a unitary operator U on $L_2 = L_2(X, \mathfrak{F}, \mu)$, a function $f \in L_2$, and a nonnull set A in $\mathfrak{F}$ such that $n^{-1}\Sigma_{i=1}^n U^i f$ diverges on A.

The following remarks on dilations of contractions are quite simple, but nevertheless they seem to us worth making. In particular, since, by a result of D. L. Burkholder [6], the pointwise ergodic theorem for nonpositive contractions in L_2 fails, Proposition 1 seems of interest.

The oldest and best known dilation theorem, due to Sz. Nagy (see, e.g., [7]), asserts that if T is a contraction on a Hilbert space H, then there exists a larger Hilbert space $K = H \oplus H'$ and a unitary operator U in K so that

$$T^n f = PU^n f$$

for each integer $n \geq 0$ and for each $f \in H$, where $P: K \longrightarrow H$ is the projection from K to H. We first observe that the following version of Nagy's theorem is also true.

Remark 1. If $H = L_2(X, \mathfrak{F}, \mu)$ then K may be chosen to be $K = L_2(Y, \mathfrak{G}, \nu)$, where $Y \supset X$, $\mathfrak{G} \supset \mathfrak{F}$ (in particular $X \in \mathfrak{G}$) and ν is an extension of μ.

The proof is the same as the one given, e.g., in [7, p. 16]: One only chooses each component Hilbert space to be an L_2 space. If H and K are related to each other as in Remark 1, then the projection $P: K \longrightarrow H$ will be called a *geometric* projection and the associated dilation will be called a *geometric* dilation. A geometric projection preserves pointwise convergence since it is just the restriction of an L_2 function to a smaller part of the domain of definition. Consequently Remark 1, combined with Burkholder's result [6, p. 128] that there are contractions of L_2 for which the pointwise ergodic theorem fails, gives the following result.

Proposition 1. *Let $(X, \mathfrak{F}, \mu)$ be a nonatomic measure space. There exists a unitary operator U on $L_2 = L_2(X, \mathfrak{F}, \mu)$, a function $f \in L_2$ and a nonnull set $A \in \mathfrak{F}$ so that $n^{-1}\Sigma_{i=1}^n U^i f$ diverges on A.*

Received by the editors August 15, 1974.
AMS (MOS) subject classifications (1970). Primary 47A35.
Key words and phrases. L_p-space, contraction, isometry, dilation.
[1] Research supported by NRC Grant A3974.
[2] Research supported by NSF Grant GP34118.

Recently dilation theorems were obtained [1], [2], [5] for positive contractions T on L_p spaces, $1 \leq p < \infty$. In these theorems the dilated space is also an L_p-space, U is a positive (but not necessarily invertible) isometry, and P is a conditional expectation E. Since the conditional expectation does not preserve pointwise convergence, it would be desirable to replace E by a geometric projection. It is, however, observed below that a large class of L_p contractions do not admit of geometric dilations.

Proposition 2. *Let* $1 \leq p < \infty$. (i) *If* T *is a contraction* (*or only a bounded linear operator*) *on the* L_p *space of a Borel space* $(X, \mathfrak{F}, \mu)$, *and if* $T\chi_A \cdot T\chi_B = 0$ *whenever* A *and* B *are disjoint sets in* $\mathfrak{F}$, *then* T *is of the form* $Tf = h(f \circ \phi)$, *where* $h \in L_p$ *and* ϕ *is a measurable point-transformation. Let* T *be a contraction on* L_p. *Assume that there exist disjoint sets* A, B *in* $\mathfrak{F}$ *such that* $T\chi_A \cdot T\chi_B \neq 0$. (ii) *If* $p \neq 2$, *then* T *does not have a geometric dilation to an isometry.* (iii) *If* $p = 2$, *then* T *does not have a geometric dilation to a positive isometry.*

Proof. The proof of (i) is similar to the proof of Banach's theorem on representation of L_p isometries, $p \neq 2$ (see, e.g., [9, p. 333]). Case (ii) follows from Banach's theorem and the fact that a point-transformation preserves disjointness of sets. To prove case (iii), it is enough to note that Banach's result remains true for *positive* isometries on L_2, as was observed by A. Ionescu Tulcea [8].

Proposition 2 implies that T can be dilated geometrically to an isometry (to a positive isometry if $p = 2$) only if T is already induced by a point-transformation.

For applications of the dilation theorem to the maximal ergodic theorem see [1], [4] and [3]. In this connection, we note that in [4] there is a technical error just before formula (7). It should be clear, however, that this can be corrected trivially without changing this formula or any of the results in [4]: While the supremum need not be reached, there exist mutually disjoint sets E_n such that $S(T)f$ is arbitrarily close to $|\Sigma_{n=1}^{\infty} 1_{E_n} A_n(T)f|$, and this is sufficient.

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