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FLOW-INVARIANT DOMAINS OF HÖLDER CONTINUITY FOR NONLINEAR SEMIGROUPS[1]

ANDREW T. PLANT

ABSTRACT. Let $S(t)$ be a nonlinear semigroup, on Banach space X, generated by an accretive set A. The set of x in X such that $t \to S(t)x$ is Hölder continuous, with Hölder exponent $\sigma \in (0, 1]$, is flow-invariant and is characterised by the behaviour of the map $\lambda \to (I + \lambda A)^{-1}x$ at $\lambda = 0$.

0. **Introduction.** Let D be a subset of Banach space X, and $\bar{D}$ its closure. Let $S(t)$ be a strongly continuous semigroup on $\bar{D}$. That is S: $[0, \infty) \times \bar{D} \to \bar{D}$, $S(t+s)x = S(t)S(s)x$ and $S(t)x \to S(0)x = x$ as $t \to 0$. Suppose further

$$\|S(t)x - S(t)y\| \leq e^{\omega t}\|x - y\|. \tag{0.1}$$

That is $S \in Q_\omega(\bar{D})$ [4]. For $0 < \sigma \leq 1$ define

$$D^\sigma = \left\{ x \in \bar{D}: \limsup_{t \to 0} t^{-\sigma}\|x - S(t)x\| < \infty \right\} \tag{0.2}$$

then clearly D^σ is flow-invariant, and for each $x \in D^\sigma$, $T < \infty$, there exists $K < \infty$ such that $\|S(t)x - S(s)x\| \leq K|t - s|^\sigma$ for $0 \leq s, t \leq T$.

It is the purpose of this paper to characterise D^σ by the behaviour of the infinitesimal generator of S. The result, Theorem II, is quite natural and extends some results of Crandall [3] where the generalized domain D^1 is considered.

1. **Preliminaries.** Let X be a Banach space, and $\mathfrak{A}(\omega)$ denote the set of subsets A of $X \times X$ such that $A + \omega I$ is accretive. Let $D = D(A) = \{x: Ax \neq \emptyset\}$ and $J_\lambda = (I + \lambda A)^{-1}$. Accretive sets are defined in [4] where the following generation theorem is proved.

Theorem I. *Let* $A \in \mathfrak{A}(\omega)$. *Suppose there exists* $\lambda_0 > 0$ *such that* $\lambda_0\omega < 1$ *and*

$$R(I + \lambda A) \supset \bar{D} \quad \textit{for } 0 < \lambda \leq \lambda_0, \tag{1.1}$$

then

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$$S(t)x = \lim_{n\to\infty} J^n_{t/n} x \tag{1.2}$$

exists for $x \in \overline{D}$, *and moreover* $S \in Q_\omega(\overline{D})$.

Let (x, x^*) denote the value of $x^* \in X^*$ at $x \in X$, and $F(x) = \{x^* \in X^*: (x, x^*) = \|x\|^2 = \|x^*\|^2\}$. For $x, y \in X$ define $\langle x, y\rangle_+ = \sup\{\mathrm{Re}(x, y^*): y^* \in F(y)\}$. Some properties of this function and the corresponding $\langle x, y\rangle_-$ are given in [4, Lemma 2.16] and [5, p. 74]. In particular for $x, y \in X$ and $\alpha \in \mathbf{R}$

$$|\langle x, y\rangle_+| \le \|x\|\|y\|, \qquad \langle x + \alpha y, y\rangle_+ = \langle x, y\rangle_+ + \alpha\|y\|^2. \tag{1.3}$$

Our proof of Theorem II in the next section is based on the following result.

Lemma 1.1. *Let* A *and* $S(t)$ *be as in Theorem* I. *Let* $x \in \overline{D}$ *and* $[x_0, y_0] \in A$ *then*

$$\|x_0 - S(t)x\|^2 - \|x_0 - x\|^2 \le 2\int_0^t \langle y_0 + \omega(x_0 - S(\tau)x), x_0 - S(\tau)x\rangle_+ \, d\tau. \tag{1.4}$$

The proof of (1.4) for the case $\omega = 0$ is given by [7, Equation (2.10)]. The case for general ω is easy to deduce from [5, Equation (3.8)]. Replacing x by $S(s)x$ gives

$$H(t) = \int_0^t \langle y_0 + \omega(x_0 - S(\tau)x), x_0 - S(\tau)x\rangle_+ \, d\tau - \tfrac{1}{2}\|x_0 - S(t)x\|^2 \tag{1.5}$$

is nondecreasing.

2. Hölder continuous domains. We characterise D^σ by defining the following two functions on $\overline{D}$.

$$|Sx|_\sigma = \limsup_{t\to 0} t^{-\sigma}\|x - S(t)x\|, \qquad |Ax|_\sigma = \limsup_{\lambda\to 0} \lambda^{-\sigma}\|x - J_\lambda x\|.$$

Theorem II. *Suppose* $A \in \mathfrak{A}(\omega)$ *and* (1.1) *holds. Let* $S(t)$ *be defined by* (1.2). *Then for* $0 < \sigma \le 1$,

$$(1/3)|Sx|_\sigma \le |Ax|_\sigma \le 3|Sx|_\sigma, \tag{2.1}$$

$$D^\sigma = \{x \in \bar{D}: |Sx|_\sigma < \infty\} = \{x \in \bar{D}: |Ax|_\sigma < \infty\}. \tag{2.2}$$

Proof. Clearly (2.2) is a consequence of (2.1). To avoid considering separate cases we may assume in full generality that $\omega > 0$. Let $\lambda > 0$ and $\lambda\omega < 1$, and in (1.5) set $x_0 = J_\lambda x$, $y_0 = \lambda^{-1}(x - x_0)$. If $z(\tau) = \|x_0 - s(\tau)x\|$ and $f(\tau) = \lambda^{-1}[\|x - S(\tau)x\| - (1 - \lambda\omega)z(\tau)]$ then by (1.3)

$$\begin{aligned}
&\lambda\langle y_0 + \omega(x_0 - S(\tau)x), x_0 - S(\tau)x\rangle_+ \\
&\qquad = \langle x - S(\tau)x - (1 - \lambda\omega)(x_0 - S(\tau)x), x_0 - S(\tau)x\rangle_+ \\
&\qquad = \langle x - S(\tau)x, x_0 - S(\tau)x\rangle_+ - (1 - \lambda\omega)z(\tau)^2 \le \lambda f(\tau)z(\tau).
\end{aligned}$$

Therefore $\int_0^t f(\tau)z(\tau)\,d\tau - \frac{1}{2}z(t)^2$ is nondecreasing. Consequently, if $z(t) \neq 0$, $0 \leq f(t) - Dz(t)$, where D represents any of the four Dini derivates. However, if $D^- z(t)$ (resp. $D_- z(t)$) represents the upper (resp. lower) *left-hand* derivate of z at $t > 0$, then $z(t) = 0$ implies $D_- z(t) \leq D^- z(t) \leq 0 \leq f(t)$. Consequently for all $t > 0$, $0 \leq f(t) - D_- z(t) = D^-[\int_0^t f(\tau)d\tau - z(t)]$; and so by a classical result, e.g. [8, p. 84], the bracketed term is nondecreasing. In particular $z(t) - z(0) \leq \int_0^t f(\tau)\,d\tau$.

Now set $x(t) = \|x - S(t)x\|$, $y(\lambda) = \|x - J_\lambda x\| = \|x - x_0\| = z(0)$. Then $|x(t) - y(\lambda)| \leq z(t)$ and consequently

$$|x(t) - y(\lambda)| - y(\lambda) \leq \lambda^{-1} \int_0^t x(\tau) - (1 - \lambda\omega)|x(\tau) - y(\lambda)|\,d\tau. \tag{2.3}$$

The modulus signs may be removed to give four inequalities of which one is trivial. We use two of the remaining three. First

$$x(t) - 2y(\lambda) \leq \lambda^{-1} \int_0^t \lambda\omega x(\tau) + (1 - \lambda\omega)y(\lambda)\,d\tau$$

which, after integrating the constant term and applying Gronwall's lemma [6, p. 283], gives

$$x(t) \leq [2 + (1 + \lambda\omega)(\exp(\omega t) - 1)/\lambda\omega]y(\lambda). \tag{2.4}$$

Now set $\lambda = t$, divide by t^σ and let $t \to 0$ to obtain the first inequality in (2.1). Returning to (2.3),

$$-x(t) \leq \lambda^{-1} \int_0^t (2 - \lambda\omega)x(\tau) - (1 - \lambda\omega)y(\lambda)\,d\tau$$

which rearranges to

$$(1 - \lambda\omega)y(\lambda) \leq (\lambda/t)x(t) + (2 - \lambda\omega)t^{-1} \int_0^t x(\tau)\,d\tau. \tag{2.5}$$

For the second inequality in (2.1) we may assume $|Sx|_\sigma < \infty$, so $L(t) = \sup\{\tau^{-\sigma}x(\tau)\colon 0 < \tau \leq t\} \to |Sx|_\sigma$ as $t \to 0$ and $\int_0^t x(\tau)\,d\tau \leq (1 + \sigma)^{-1}L(t)t^{1+\sigma}$. Then setting $\lambda = t$ in (2.5) gives

$$(1 - t\omega)y(t) \leq x(t) + (2 - t\omega)(1 + \sigma)^{-1}L(t)t^\sigma$$

and the required inequality follows. This completes the proof.

From (2.4) and (2.5) we easily deduce global estimates:

Corollary. *If* $\rho(t) = (e^t - 1)/t$ $(t > 0)$, $\rho(t) = 1$ $(t \leq 0)$, $\omega^+ = \max\{\omega, 0\}$ *and* $\lambda\omega < 1$ *then*

$$\|x - S(t)x\| \leq [2 + (1 + \lambda\omega^+)\rho(\omega t)t/\lambda]\|x - J_\lambda x\|, \tag{2.6}$$

$$(1 - \lambda\omega^+)\|x - J_\lambda x\| \leq (\lambda/t)\|x - S(t)x\| + (2 - \lambda\omega^+)t^{-1} \int_0^t \|x - S(\tau)x\|\,d\tau. \tag{2.7}$$

In particular if $\omega = 0$

$$\frac{1}{3}\|x - S(t)x\| \leq \|x - J_t x\| \leq \|x - S(t)x\| + \frac{2}{t}\int_0^t \|x - S(\tau)x\|\,d\tau. \tag{2.8}$$

Remark 1. The first inequality in (2.8) was announced by D. Brézis [2] for the case X is a Hilbert space. The general case has the following simple proof, for which we thank the referee:

$$\|x - S(t)x\| \le \|x - J_t x\| + \|J_t x - S(t)J_t x\| + \|S(t)J_t x - S(t)x\|$$

$$\le 2\|x - J_t x\| + \|J_t x - S(t)J_t x\|,$$

but

$$\|J_t x - S(t)J_t x\| \le t\|A_t x\| = \|x - J_t x\|.$$

The second inequality in (2.8) should be compared with $\|x - J_t x\| \le 3\|x - S(t)x\|$ obtained by Brézis for the case X is a Hilbert space and A is a gradient. In the general case periodic orbits are possible, and the Brézis estimate fails. For example consider the rotation group in $\mathbf{R}^2$.

Finally, if X is a Hilbert space, one deduces from Theorem 4 of [2] with $p = \infty$ that

$$\frac{1}{3} \le \left(\sup_{0<t\le 1} t^{-\sigma}\|x - J_t x\|\right) \Big/ \left(\sup_{0<t\le 1} t^{-\sigma}\|x - S(t)x\|\right) \le 6.$$

(2.8) gives the same result with 6 replaced by $(3 + \sigma)/(1 + \sigma) \le 3$.

Remark 2. Our method can easily be used to improve estimates (2.1). In (2.4), (2.5) set $\lambda = \alpha t$, take the limit as before and then minimize with respect to $\alpha > 0$. This gives

$$\gamma(\sigma)|Sx|_\sigma \le |Ax|_\sigma \le [(1 + \sigma)^{1-\sigma}\gamma(\sigma)]^{-1}|Sx|_\sigma \tag{2.9}$$

where $\gamma(\sigma) = \sigma^\sigma(\tfrac{1}{2}(1 - \sigma))^{1-\sigma}$ $(\sigma < 1)$, $\gamma(1) = 1$, which is best possible at $\sigma = 1$ (i.e. $|Sx|_1 = |Ax|_1$) (proved by Crandall [3]). In the next section we give an example where $|Ax|_{1/2} < |Sx|_{1/2}$.

Remark 3. The results in this section have wider applicability than we have so far indicated. Condition (1.5) expresses the fact that $S(t)x$ is, in the terminology of Ph. Bénilan [1], a *solution intégrale* of $(du/dt) + Au \ni 0$. Consequently our results apply to such solutions, provided $x \in R(I + \lambda A)$.

3. **An example.** We consider an example of Webb [9]. Let X be the Banach space of bounded uniformly continuous real-valued functions on $[0, \infty)$ with supremum norm. Let $Af = -f'$, $D = D(A) = \{f: f' \in X\}$. Then A is closed, linear, densely defined and m-accretive. The corresponding semigroup is translations, $(S(t)f)(s) = f(s + t)$, so D^σ is the subspace of uniformly Hölder continuous functions with Hölder exponent σ. Moreover

$$(J_\lambda f)(t) = \lambda^{-1}\int_t^\infty f(s)\exp[(t - s)/\lambda]\,ds.$$

Now set $f(t) = (1 - t)^{1/2}$ $(t < 1)$, $f(t) = 0$ $(t \ge 1)$. So $|Sf|_{1/2} = 1$, and it is an easy computation to show

$$\lambda^{-1/2}(f(t) - (J_\lambda f)(t)) = \int_0^y \exp(x^2 - y^2)\,dx \qquad (t < 1)$$
$$= 0 \qquad (t \geq 1)$$

where $y = (1-t)^{1/2}\lambda^{-1/2}$. Therefore

$$\lambda^{-1/2}\|f - J_\lambda f\| = \sup_y \int_0^y \exp(x^2 - y^2)\,dx$$

where sup is taken over the range $0 \leq y \leq \lambda^{-1/2}$. Thus

$$|Af|_{1/2} = \sup_{0 \leq y} \int_0^y \exp(x^2 - y^2)\,dx.$$

But

$$\int_0^y \exp(x^2 - y^2)\,dx \leq e^{-y^2}\int_0^y e^{xy}\,dx = (1 - e^{-y^2})/y < (2/e)^{1/2}.$$

Therefore

$$|Af|_{1/2} \leq (2/e)^{1/2} < 1 = |Sf|_{1/2}.$$

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REFERENCES

1. Ph. Bénilan, *Solutions intégrales d'equations d'évolution dans un espace de Banach*, C. R. Acad. Sci. Paris Sér. A-B 274 (1972), A47–A50. MR 45 #9212.

2. D. Brézis, *Classes d'interpolation associées à un opérateur monotone*, C. R. Acad. Sci. Paris Sér. A-B 276 (1973), A1553–A1556.

3. M. G. Crandall, *A generalized domain for semigroup generators*, M.R.C. Technical Report #1189, University of Wisconsin, Madison, Wis.

4. M. G. Crandall and T. M. Liggett, *Generation of semi-groups of nonlinear transformations on general Banach spaces*, Amer. J. Math. 93 (1971), 265–298. MR 44 #4563.

5. M. G. Crandall and A. Pazy, *Nonlinear evolution equations in Banach spaces*, Israel J. Math. 11 (1972), 57–94. MR 45 #9214.

6. J. Dieudonné, *Fondements de l'analyse moderne*, Pure and Appl. Math., vol. 10, Academic Press, New York, 1960. MR 22 #11074.

7. I. Miyadera, *Some remarks on semi-groups of nonlinear operators*, Tôhoku Math. J. (2) 23 (1971), 245–258. MR 45 #5805.

8. H. L. Royden, *Real analysis*, Macmillan, New York 1963. MR 27 #1540.

9. G. F. Webb, *Continuous nonlinear pertubations of linear accretive operators in Banach spaces*, J. Functional Analysis 10 (1972), 191–203.

FLUID MECHANICS RESEARCH INSTITUTE, UNIVERSITY OF ESSEX, COLCHESTER CO4 3SQ, ENGLAND