PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 53, Number 1, November 1975

METRIC RIGIDITY IN E^n

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ABSTRACT. In answer to a question rasied by L. Janos, it is shown (i) that E^n is not a finite union of metrically rigid subsets; (ii) that if $2^\omega > \kappa^+$, then E^n is not the union of κ metrically rigid subsets, where κ is an infinite cardinal; and (iii) that if $2^\omega = \kappa^+$, then E^1 is the union of κ metrically rigid subsets, and hence that E^1 is a countable union of metrically rigid subsets iff the continuum hypothesis holds. ($A \subseteq E^n$ is metrically rigid iff no two distinct two-point subsets of A are isometric.) Open question: assuming the continuum hypothesis, can E^n be written as a countable union of metrically rigid subsets if $n > 1$?

In [2], L. Janos defined the notion of metric rigidity as follows:

Definition. Let (X, d) be a metric space. A subset $A \subseteq X$ is *metrically rigid* (m.r.) with respect to d iff no two distinct two-point subsets of A are isometric.

Since the concept is evidently very much metric-dependent (in the referee's words, with which the authors agree, it is ''truly nonsense''), two natural kinds of questions arise:

(a) What sorts of metrically rigid subsets can be found in well-known spaces having ''standard'' metrics, such as E^n?

(b) What metrizable spaces admit rigid metrics?

A partial answer to (b) is announced in [1], as follows:

Theorem 1 (N. Passell). *If X is metrizable,* $\operatorname{card}(X) \leq 2^\omega$, *and X can be represented as a countable product of discrete spaces, then X admits a rigid metric. So, in particular, the Cantor set admits a rigid metric. (The first author has, as a curiosity, succeeded in producing an explicit rigid metric on the Cantor set, explicit in the sense that, given any two real numbers in the usual ''middle-thirds'' Cantor set, the distance between them can be computed to any desired degree of accuracy.)*

Corollary 2. *If X is separable and metrizable, then X admits a rigid metric iff X is 0-dimensional.*

Proof. This follows immediately from the proof of the Urysohn metriza-

Received by the editors October 28, 1974.
AMS (MOS) subject classifications (1970). Primary 57A15.
Key words and phrases. Metric rigidity, rigid sets, continuum hypothesis.

tion theorem and the fact that if X is 0-dimensional, then X has a clopen base.

In this paper, however, we shall be concerned primarily with questions of type (a). Specifically, we investigate the number of (disjoint) metrically rigid subsets (with respect to the usual Pythagorean metric) needed to cover E^n, Euclidean n-dimensional space. We employ the usual conventions that an ordinal *is* the set of smaller ordinals, and that a cardinal *is* an initial ordinal, and we denote by κ^+ the least cardinal greater than the cardinal κ. We reserve the letters κ and λ for infinite cardinals, while α, β, γ, and δ will be used to denote ordinals.

Theorem 3. *E^1 is not a finite union of m.r. sets.*

Proof. Suppose that E^1 is the union of n m.r. sets, $R_1, \ldots, R_n$, and let f: $E^1 \to \{1, \ldots, n\}$ take each real number x to the (unique) $i \in \{1, \ldots, n\}$ such that $x \in R_i$. Let $\mathfrak{B} = \{B_i\colon 0 \leq i \leq n^{n+1}\}$ be any family of $n^{n+1} + 1$ pairwise disjoint blocks of $n + 1$ consecutive integers, say $B_i = \langle m_i, \ldots, m_i + n\rangle$. Let $F(B_i) = \langle f(m_i), \ldots, f(m_i + n)\rangle$; then there must be integers i and j such that $0 \leq i < j \leq n^{n+1}$ and $F(B_i) = F(B_j)$. Similarly, there must be integers k and p such that $0 \leq k < p \leq n$ and $f(m_i + k) = f(m_i + p) = f(m_j + k) = f(m_j + p) = a$, say, for some a with $a \in \{1, \ldots, n\}$, and it follows at once that R_a is not m.r., a contradiction.

Corollary 4. *No E^n, for $n > 0$, is a finite union of m.r. subsets.*

Proof. E^1 embeds isometrically in E^n.

Theorem 5. *If $2^\omega > \kappa^+$, then E^n is not the union of κ m.r. subsets. In particular, if the continuum hypothesis is false, then E^n is not the countable union of m.r. sets.*

Proof. As in Theorem 3, it suffices to consider the case $n = 1$. Suppose that $E^1 = \bigcup\{R_\alpha\colon \alpha < \kappa\}$, where the R_α's are pairwise disjoint and m.r. Let S be an arbitrary subset of E^1 of power κ^+. If $A \subseteq E^1$, define $D(A)$ to be the set of nonzero distances occurring between members of A, and note that if A is infinite, then $\operatorname{card}(D(A)) = \operatorname{card}(A)$. We construct a family of κ^{++} pairwise disjoint translates of S as follows.

Let $r_0 = 0$. If $\alpha < \kappa^{++}$ and r_β has been defined for all $\beta < \alpha$, choose $r_\alpha \in E^1 \backslash \{r_\beta \pm a\colon \beta < \alpha \ \&\ a \in D(S)\}$; this is possible since at each stage there are 2^ω valid choices. Let $S_\alpha = \{x + r_\alpha\colon x \in S\}$; clearly the S_α's are disjoint.

Now, for each $\alpha < \kappa^{++}$ there is $f(\alpha) < \kappa$ and $T_\alpha \subseteq S_\alpha \cap R_{f(\alpha)}$ such that $\operatorname{card}(T_\alpha) = \kappa^+$. Similarly, there is $\alpha_0 < \kappa$ such that $\operatorname{card}(f^{-1}(\alpha_0)) = \kappa^{++}$, so we may assume that for all $\alpha < \kappa^{++}$, $T_\alpha \subseteq R_{\alpha_0}$. However, $D(T_\alpha) \subseteq D(S)$ for all $\alpha < \kappa^{++}$, and $\kappa^+ = \operatorname{card}(D(T_\alpha)) \leq \operatorname{card}(D(S)) \leq \kappa^+$, so again by the pigeon-hole principle, there are $a \in D(S)$ and $\alpha < \beta < \kappa^{++}$ such that $a \in D(T_\alpha) \cap D(T_\beta)$; and since $T_\alpha \cap T_\beta = \varnothing$, this contradicts the metric rigidity of R_{α_0}.

The foregoing results are, of course, negative; before proving a (partial) positive result we introduce some additional notation.

Definition. If (X, d) is a metric space with $S \subseteq X$, we define

$$\mathrm{cl}_M(S) = \{x \in X \colon \exists y_1, y_2 \in S(y_1 \neq y_2 \;\&\; d(y_1, x) = d(x, y_2))\};$$

$$\mathrm{cl}_C(S) = \{x \in X \colon \exists y \in S,\ a \in D(S)\ (d(x, y) = a)\};$$

$$\mathrm{cl}^1(S) = S \cup \mathrm{cl}_M(S) \cup \mathrm{cl}_C(S);$$

$$\mathrm{cl}^{n+1}(S) = \mathrm{cl}^1(\mathrm{cl}^n(S));$$

$$\mathrm{CL}(S) = \bigcup \{\mathrm{cl}^n(S) \colon n = 1, 2, \ldots\}.$$

Theorem 6. *Let* (X, d) *be a metric space and* κ *a cardinal such that*

(a) $\mathrm{card}(X) = \kappa^+$;

(b) *If* x *and* y *are distinct points of* X, *then there are at most* κ *points* $z \in X$ *such that* $d(x, z) = d(z, y)$; *and*

(c) *if* $x \in X$ *and* $a > 0$, *then there are at most* κ *points* $z \in X$ *such that* $d(x, z) = a$.

Then X *is the union of* κ *m.r. subsets.*

Proof. Write $X = \{x_\alpha \colon \alpha < \kappa^+\}$, and for $\alpha < \kappa^+$ define $X_\alpha = \mathrm{CL}(\{x_\beta \colon \beta \leq \alpha\})$; then conditions (b) and (c) of the theorem guarantee that $\mathrm{card}(X_\alpha) \leq \kappa$. For $\alpha < \kappa^+$ define $D_\alpha = X_\alpha \setminus \bigcup \{X_\beta \colon \beta < \alpha\}$, and let $\mathfrak{D} = \{D_\alpha \colon \alpha < \kappa^+\}$, so that $\mathfrak{D}$ is a partition of X into sets, possibly empty, of power at most κ. For each $\alpha < \kappa^+$ there is a 1-1 function f_α mapping D_α into κ, and, hence, we may set $f = \bigcup \{f_\alpha \colon \alpha < \kappa^+\}$ and $R_\alpha = f^{-1}(\alpha)$.

Clearly $\{R_\alpha \colon \alpha < \kappa\}$ is a partition of X, and it only remains to be shown that each R_α is m.r. This follows from the definition of CL by an examination of the ways in which R_α can fail to be m.r. and is entirely straightforward.

Corollary 7. E^1 *is the countable union of m.r. subsets iff the continuum hypothesis holds.*

Proof. Necessity is a consequence of Theorem 5. Sufficiency follows from the observation that E^1 satisfies (b) and (c) of Theorem 6 when $\kappa = \omega$, and, moreover, satisfies (a) under the assumption of the continuum hypothesis.

It is perhaps worth noting that in the special case in which $\kappa = \omega$ and $X = E^1$, an alternate proof of Theorem 6 can be obtained by viewing the real numbers as an ω_1-dimensional vector space over the field of rational numbers; this approach gives slightly more insight into the structure of the rigid sets obtained. However, neither approach generalizes readily to E^n for $n > 1$, and we do not know whether the continuum hypothesis implies that E^n is a countable union of m.r. sets if $n > 1$.

The foregoing results concern minimal m.r. covers of E^n. At the other extreme we note that the methods of [3] can be applied to partition E^n into 2^ω m.r. sets, each of power 2^ω in every nonempty, open subset of E^n.

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