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AN ALTERNATE CHARACTERIZATION OF THE CANTOR SET

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ABSTRACT. Let X be a compact metric space such that, up to homeomorphism, X has only two nonempty open subsets. Then X is homeomorphic to the Cantor discontinuum.

It is well known that any compact, perfect, totally disconnected metric space is homeomorphic to the Cantor "middle thirds" set K. The Cantor set is also known to have the following property: Up to homeomorphism, K has only two nonempty open subsets [1]. We will show that, among compact metric spaces, this property characterizes K.

Definition. A metric space X *has property* W if:

(a) X has at least one nonempty compact open subset and at least one noncompact open subset (one of these may be X).

(b) Any two nonempty compact open subsets of X are homeomorphic.

(c) Any two noncompact open subsets of X are homeomorphic.

Theorem. *Let X be a compact metric space. Then X is homeomorphic to K if and only if X has property W.*

Proof. The property is preserved by homeomorphism, and is thus necessary. Suppose now that X has property W. Any isolated point of X would, by (b), be homeomorphic to X, making X a one-point space. This would contradict (a), so X is perfect. We now show that X is disconnected.

Let x and y be distinct points of X, d the distance from x to y, U_x and U_y the open balls of radius $d/3$ about x and y respectively, and $U = U_x \cup U_y$. If U is homeomorphic to X, then X is disconnected. If U is not homeomorphic to X, then U is noncompact. Since X is perfect, we have for each $x \in X$ that $X\setminus\{x\}$ is noncompact and open, therefore (being homeomorphic to U) disconnected. Thus every point of X is a cut-point of X. Here too X must be disconnected, as every metric continuum has at least two non-cut-points.

Finally, take a point $x \in X$ and consider the quasicomponent (equals the component) of x, say C. The set C cannot be open, for then it would be homeomorphic to X and disconnected. Thus $X\setminus C$ is not compact and is

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homeomorphic to $X\setminus\{x\}$. Now each $y \in X\setminus C$ has a compact neighborhood $V \subset X\setminus C$ (the complement of a closed-and-open $U \subset X$ such that $x \in U$ and $y \notin U$). The same holds for $X\setminus\{x\}$, which shows that X is totally disconnected. As a compact, perfect, totally disconnected metric space, X is homeomorphic to K.

Corollary. *Let X be a noncompact metric space. Then X is homeomorphic to $K\setminus\{0\}$ if and only if X has property W.*

The proof is trivial.

REFERENCE

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