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ON THE CENTER OF SOME FINITE LINEAR GROUPS

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ABSTRACT. This note proves two results, one in characteristic p and the other in characteristic zero, which restrict the order of the center of some finite linear groups of degree less than a prime p which divides the group order.

G denotes a finite group, p a fixed odd prime, P a Sylow p-subgroup of G. Z is the center of G and $z = |Z|$.

Theorem 1. *Assume $G = G'$, G is not of type $L_2(p)$, P is cyclic, and for some field K of characteristic p, there is a faithful, indecomposable KG-module L of dimension $d < p$. If d is odd then $d \geq (z/(z+2))p$. If d is even then*

$$\begin{aligned} d &\geq (z/(z+2))(p+1) \qquad (z \text{ odd}) \\ &\geq (z/(z+4))(p+1) \qquad (z \text{ even}). \end{aligned}$$

Theorem 2. *Assume $G = G'$, P has order p and is not normal in G, the number t of conjugate classes of p-elements of G is at least 3, and G has a faithful irreducible complex character χ of degree $d < p - 1$. Let $e = (p-1)/t$. Then $z \leq 2d/(e+1)$.*

Remarks. (i) Theorem 1 supplements [1, Theorem 5.11]. While the fractional multiples of p given in [1, Theorem 5.11] are a little better than in Theorem 1, an annoying (especially for large values of z) remainder term in the earlier result is dispensed with here. One consequence is the following: It is known that $z|d$ under the hypotheses of Theorem 1. As a corollary of the theorem, we have that if $z = d$, then $d \geq p - 3$.

(ii) Theorem 2 is proved by exploiting the methods of [2], which were themselves a variation on those of [8]. The numerical case $p = 31$, $d = z = 28$, $e = 3$ listed in [1, §8], and not ruled out by previous results, is eliminated by Theorem 2. For in that case the modular representation involved in [1] lifts to an ordinary representation to which Theorem 2 can be applied.

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(iii) Apparently, no groups are known which satisfy either the hypotheses of Theorem 1 with $p \geq 13$ and $d < p - 2$, or the hypotheses of Theorem 2 with $p > 7$ and $d < p - 2$. If such groups do exist, their p-local structure is quite restricted, as our results indicate.

Proof of Theorem 1. Since L_P is indecomposable [7], and remains indecomposable under all field extensions (as P is cyclic), we may assume K is a splitting field for all subgroups of G. Let $d = p - s$. By [1, (5.3)], the nonprojective summands of $L \otimes L$ are L_i, $0 \leq i \leq s - 1$, of dimensions $2i + 1 + m_i p$, with $\Sigma_{i=0}^{s-1} m_i \leq p - 2s$. Now $z|d$ [1, Proposition 5.1] and by [1, (5.10)],

$$z|2(2i + 1 + m_i p), \qquad 0 \leq i \leq s - 1. \tag{1}$$

For any integer i with $0 \leq i < (s - 1)/2$, let $i' = s - 1 - i$. Then (1) implies

$$z|2(2s + (m_i + m_{i'})p), \qquad 0 \leq i < (s - 1)/2. \tag{2}$$

Suppose d is odd. Then z is odd, and (2) yields

$$(m_i + m_{i'})p \equiv -2s \pmod z, \qquad 0 \leq i \leq s/2 - 1.$$

Since $p \equiv s \pmod z$, and $(s, z) = 1$, we have

$$m_i + m_{i'} \equiv -2 \pmod z, \qquad 0 \leq i \leq s/2 - 1.$$

Thus $m_i + m_{i'} \geq z - 2$, $0 \leq i \leq s/2 - 1$, so that

$$p - 2s \geq \sum_{i=0}^{s-1} m_i \geq \sum_{i=0}^{s/2-1} (m_i + m_{i'}) \geq (s/2)(z - 2).$$

It follows that $s \leq (2/(z + 2))p$, whence $d \geq (z/(z + 2))p$.

Suppose d is even. If z is odd, again we have

$$m_i + m_{i'} \equiv -2 \pmod z, \qquad 0 \leq i < (s - 1)/2.$$

Also, $z|2(s + pm_{(s-1)/2})$ implies $m_{(s-1)/2} \equiv -1 \pmod z$. Thus

$$p - 2s \geq m_{(s-1)/2} + \sum_{i=0}^{(s-3)/2} (m_i + m_{i'}) \geq z - 1 + (z - 2)(s - 1)/2.$$

It follows that $s \leq (2/(z + 2))p - z/(z + 2)$, hence

$$d \geq (z/(z + 2))p + z/(z + 2) = (z/(z + 2))(p + 1). \tag{3}$$

If z is even, (2) implies the above congruences are still valid modulo $z/2$. So we may replace z by $z/2$ in (3) to obtain $d \geq (z/(z + 4))(p + 1)$.

Proof of Theorem 2. Assume the hypotheses of Theorem 2. Then $d = p - e$ [5]. If $p = 7$, no such groups exist [6], so we may assume $p > 7$. Feit's reduction argument [8, (6.1)] shows that G is not of type $L_2(p)$. Then by [8, (2.1)], G satisfies conditions $(*)$ of [8], and $z|d$. If $e = 1$ then $d = p - 1$, a contradiction. If $e = 2$ then $G \approx \mathrm{SL}_2(2^a)$ and $z = 1$ [9]. So we may assume $2 < e < (p-1)/2$. Also, [8, Theorem 1] implies z is even, hence e is odd.

Let F be a p-adic number field with ring of integers R such that F and $R/J(R) = K$ are splitting fields for all subgroups of G. Let M be an R-free RG-module such that $M \otimes_R F$ affords χ and $L = M/J(R)M$ is indecomposable [10]. $P \ntriangleleft G$ implies L is faithful, so the situation of Theorem 1 holds with $s = e$.

Let N be the normalizer of P in G, and let $L_N = V_{p-e}(\lambda)$ (cf. [1, §5]), where λ is a linear character: $N \to K$. Let L_i be as above, $0 \leq i \leq e-1$. L_i has Green correspondent $V_{2i+1}(\lambda^2\alpha^{e+i})$ [1, §5]. Let $\chi_Z = \chi(1)\eta$, η a faithful linear character: $Z \to F$. Now $\eta(Z) \subseteq R$, and if $\bar{\eta}$ denotes η composed with the canonical homomorphism: $R \to K$, then $\bar{\eta} = \lambda_Z$. There is a one-one correspondence between the p-blocks of positive defect and the distinct powers of η: an irreducible character ζ of G is in block B_n if and only if $\zeta_Z = \zeta(1)\eta^n$ [4]. Thus χ, L are in B_1 and the L_i are all in B_2 [1, §4].

Let ζ_j, $1 \leq j \leq t$, be the exceptional characters in B_2. Then $\zeta_{j_Z} = \zeta_j(1)\eta^2$. By [8, (4.1)], $\zeta_j(1) = mp + e$ for some positive integer m, independent of j.

We may assume F is sufficiently large so that for each i with $0 \leq i < (e-1)/2$, there is an R-free RG-module X_i such that $X_i/J(R)X_i \approx L_i \oplus L_{e-i-1}$ [3, Lemma 2.1]. Also, there is an R-free RG-module Y such that $Y/J(R)Y \approx L_{(e-1)/2}$. Now the ζ_j occur in the character afforded by each X_i with total multiplicity at least 2, and in the character afforded by Y with multiplicity at least 1 [3, Lemma 2.2]. Hence

$$\dim_K L \otimes L \geq \operatorname{rank}_R \left(Y \oplus \sum_{i=0}^{(e-3)/2} X_i \right) \geq 2\zeta_1(1)(e-1)/2 + \zeta_1(1) = e\zeta_1(1).$$

Therefore $(p-e)^2 \geq e(mp+e)$. It follows that $m \leq (p-2e)/e = t - 2 + 1/e$, whence $m \leq t - 2$.

Since $\zeta_{j_Z} = \zeta_j(1)\eta^2$, we set the determinant of the appropriate scalar matrices equal to 1 (as $G = G'$) to see that $\eta^{2(mp+e)} = 1$. Since η is faithful on the cyclic group Z, it follows that $z | 2(mp+e)$, hence $mp + e \equiv 0 \pmod{z/2}$. Now $z | p - e$ implies $me + e \equiv 0 \pmod{z/2}$ and $(z, e) = 1$ yields $m + 1 \equiv 0 \pmod{z/2}$. Therefore $t - 2 \geq m \geq z/2 - 1$. Hence $z \leq 2t - 2 = 2(d-1)/e < 2d/e$. Since $z|d$, we have $z \leq 2d/(e+1)$.

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