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ON ASYMPTOTIC BEHAVIORS OF ANALYTIC MAPPINGS ON THE MARTIN BOUNDARY

MIKIO NIIMURA

ABSTRACT. Some generalizations of the analogue of Collingwood and Cartwright in the large of Iversen's theorem are given.

Let f be a nonconstant analytic mapping of a hyperbolic Riemann surface R into a Riemann surface R'. Let R^* and R'^* denote the Martin compactification and any compactification of R and R', respectively. Δ and Δ' denote the Martin ideal boundary of R and the ideal boundary of R', respectively. $\bar{A}$, A^c and $\operatorname{int} A$ mean the closure, the complement and the interior of a set A ($\subset R^*$ or R'^*) with respect to R^* or R'^*, respectively. Let ∂A denote the relative boundary of A ($\subset R$ or R') with respect to R or R' and f_G the restriction of f to G ($\subset R$).

Let $\{G_n^{(e)}\}$ be a determinant sequence of Kerékjártó-Stoïlow's ideal boundary point e, and set $\Delta_e = \bigcap_n \overline{G_n^{(e)}}$ and $\Delta_{G_n^{(e)}} = \overline{G_n^{(e)}} \cap \Delta$. The cluster set of f for Δ_e is defined by $C(f, \Delta_e) = \bigcap_n \overline{f(G_n^{(e)})}$, and the range of f for Δ_e by $R(f, \Delta_e) = \bigcap_n f(G_n^{(e)})$.

In this paper we assume that the harmonic measure of Δ_e is positive.

For $b \in \Delta_1$, let F_b be a filter basis on R with respect to the fine topology, and set $f\hat{\ }(b) = \bigcap_{U \in F_b} \overline{f(U)}$. Here Δ_1 denotes the set of minimal points in Δ. If $f\hat{\ }(b)$ consists of a single point, then $f\hat{\ }(b)$ is denoted by $\hat{f}(b)$.

We say that a curve $p = \lambda(t)$ $(0 \le t < 1)$ on R converges to e, when for every n, there exists $t(n)$ such that $\lambda(t) \subset G_n^{(e)}$ for all $t \ge t(n)$. $\bigcap_{t>0} \overline{\lambda(t)}$ denotes the end of this path: $p = \lambda(t)$ $(0 \le t < 1)$. Let $\Gamma(f, \Delta_{G_n^{(e)}})$ denote the set of asymptotic points along all the paths such that the end of each path is on $\Delta_{G_n^{(e)}}$, and set $\chi(f, \Delta_e) = \bigcap_n \Gamma(f, \Delta_{G_n^{(e)}})$ and $\chi^*(f, \Delta_e) = \bigcap_n \overline{\Gamma(f, \Delta_{G_n^{(e)}})}$. If for any neighborhood V of $\alpha \in R'^*$, $V \cap \overline{\Gamma(f, \Delta_{G_n^{(e)}})}$ is a nonpolar set, we say $\alpha \in \Gamma_+(f, \Delta_{G_n^{(e)}})$ and set $\chi_*(f, \Delta_e) = \bigcap_n \Gamma_+(f, \Delta_{G_n^{(e)}})$.

Lemma 1. *If $\alpha \in \chi_*(f, \Delta_e)^c \cap C(f, \Delta_e) \cap R'$, then $\alpha \in \operatorname{int} \overline{R(f, \Delta_e)}$.*

Proof. Since $\alpha \in \chi_*(f, \Delta_e)^c \cap R'$, there exist a parametric disk V about α and $G_N^{(e)}$ such that $V \cap \overline{\Gamma(f, \Delta_{G_N^{(e)}})}$ is a polar set. Let $w = \psi(q)$

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$(q \in V)$ be a local parameter of V, and we set $\psi(V) = \{w;\ |w| < 1\}$ $(\psi(\alpha) = 0)$, $W_r = \{w;\ |w| < r,\ 0 < r < 1\}$, $C_r = \partial W_r$ and $\psi \circ f_{G_N^{(e)}} = g$.

Since $W_1 \cap \overline{\Gamma(g, \Delta_{G_N^{(e)}})}$ is a polar set, its linear measure is zero. Hence $\Gamma(g, \Delta_{G_N^{(e)}}) \cap C_r = \emptyset$ for almost all r in $0 < r < 1$. Let C_r have this property and fix r.

Since $\alpha \in C(f, \Delta_e)$, we see that $g^{-1}(W_r) \cap G_n^{(e)} \neq \emptyset$ for all $n \geq N$.

If there exists $G \in \{G_n^{(e)}\}$ such that $G \subset G_N^{(e)}$ and $g(G) \subset W_r$, then for each point b of a set H_e $(\subset \Delta_1 \cap \Delta_e)$ whose harmonic measure is positive, $\hat{g}_G(b) \in \overline{W}_r$. Indeed the harmonic measure of Δ_e is positive and g_G is a Fatou mapping of G into W_r.

Hence there exists an asymptotic path γ from a point of G to each point b of H_e such that $\lim_{p \to b; p \in \gamma} g_G(p) = \hat{g}_G(b)$. On the other hand, since $\overline{W}_r \cap \Gamma(g, \Delta_G)$ $(\supset \hat{g}_G(b)$, for $b \in H_e)$ is a polar set, the harmonic measure of H_e is zero. This is a contradiction. Thus for all $n \geq N$, we conclude that $G_n^{(e)} \cap \partial g^{-1}(W_r) \neq \emptyset$.

If $\partial g^{-1}(W_r)$ contains closed Jordan curves accumulating to e, then we see easily that $w \in R(g, \Delta_e)$ for any point w on C_r.

If for all $n \geq N$, $G_N^{(e)}$ has at least one noncompact γ_n of $\partial g^{-1}(W_r)$, let $z = \phi(p)$ be a local parameter about $p \in G_N^{(e)}$, and set $h = g \circ \phi^{-1}$. A function element $Q(w)$ of $z = h^{-1}(w)$ can be continued analytically along C_r infinitely often. Indeed if not, when w tends to a point w_1 $(\in C_r)$ along C_r, γ_n is a path whose end is on $\Delta_{G_n^{(e)}}$, and so $w_1 \in \Gamma(g, \Delta_{G_n^{(e)}})$. This is a contradiction.

Therefore any point w on C_r corresponds to an infinite number of points on γ_n for any n, and hence $w \in R(g, \Delta_e)$.

Therefore since for any point p of W_1, any neighborhood $(\subset W_1)$ of p contains points of $R(g, \Delta_e)$, we get $W_1 \subset \overline{R(g, \Delta_e)}$ and $\alpha \in \operatorname{int} \overline{R(f, \Delta_e)}$, as claimed.

Corollary 1. *If $C(f, \Delta_e)$ is nowhere dense, then $C(f, \Delta_e) \cap R' \subset \chi_*(f, \Delta_e)$.*

Proof. If $\alpha \in \operatorname{int} \overline{R(f, \Delta_e)}$, for a neighborhood V of α, any neighborhood $(\subset V)$ of any point $\beta \in V$ contains at least one point of $R(f, \Delta_e)$ and $C(f, \Delta_e)$ is not nowhere dense. Thus we have $C(f, \Delta_e) \cap R' \subset \chi_*(f, \Delta_e)$.

Lemma 2. *If $\alpha \in \chi_*(f, \Delta_e)^c \cap \chi(f, \Delta_e)^c \cap C(f, \Delta_e) \cap R'$, then $\alpha \in R(f, \Delta_e)$.*

Proof. Suppose that $\alpha \notin R(f, \Delta_e)$.

Since $\alpha \in \chi_*(f, \Delta_e)^c \cap \chi(f, \Delta_e)^c \cap R'$, there exist a parametric disk U about α and $G_N^{(e)}$ such that $U \cap \overline{\Gamma(f, \Delta_{G_N^{(e)}})}$ is a polar set and $\alpha \notin \Gamma(f, \Delta_{G_N^{(e)}})$.

All the α-points of $f_{G_N^{(e)}}$ are contained in a finite set of parametric disks $\{U_k\}$ $(k = 1, 2, \ldots, L)$ such that $U_i \cap U_j = \emptyset$ $(i \neq j)$. Let V be a parametric disk about α satisfying $V \subset (\bigcap_{k=1}^{L} f_{G_N^{(e)}}(U_k)) \cap U$. We fix r such that $\Gamma(g, \Delta_{G_N^{(e)}}) \cap C_r = \emptyset$. There exists a diameter d_r of W_r such that $\Gamma(f, \Delta_{G_N^{(e)}}) \cap d_r = \emptyset$. There exists a diameter d_r of W_r such that $\Gamma(f, \Delta_{G_N}(e)) \cap d_r = \emptyset$.

Since $b \in R(f, \Delta_e)$ for $b \in C_r$, there exists a connected component D of $g^{-1}(W_r)$ which is not relatively compact. Choose a point p on ∂D which is mapped by g to an endpoint of d_r. The function element $Q(w)$ corresponding to p can be continued analytically along d_r through the point 0 to the antipodal point and d_r is mapped on a cross-cut of D. But on the other hand, since D does not contain the zeros of g, we have a contradiction, and we conclude that $\alpha \in R(f, \Delta_e)$.

Theorem 1. *If R'^* is a metrizable and resolutive compactification of R' and, for at least one n, $\overline{\Gamma(f, \Delta_{G_n^{(e)}})}$ is a polar set, then $R(f, \Delta_e)^c \cap R' \subset \chi(f, \Delta_e)$.*

Proof. From Lemma 2, we have $R(f, \Delta_e)^c \cap R' \subset \chi_*(f, \Delta_e) \cup \chi(f, \Delta_e) \cup C(f, \Delta_e)^c$.

If $C(f, \Delta_e)^c \neq \emptyset$, there exist a parametric disk V and $G \in \{G_n^{(e)}\}$ $(G \subset G_n^{(e)})$ such that $f(G) \cap \overline{V} = \emptyset$. Since the mapping f_G of G into $R' - \overline{V}$ is a Fatou mapping, it contradicts that the harmonic measure of H_e is positive, as we see from the proof of Lemma 1.

Thus from $\Gamma_+(f, \Delta_{G_n^{(e)}}) = \emptyset$, we have $R(f, \Delta_e)^c \cap R' \subset \chi(f, \Delta_e)$.

Lemma 3. *If $\alpha \in \chi^*(f, \Delta_e)^c \cap C(f, \Delta_e) \cap R'$, then $\alpha \in \operatorname{int} R(f, \Delta_e)$.*

Proof. In Lemma 1, take "all r in $0 < r < 1$" instead of "almost all r in $0 < r < 1$" and consider "$W_1 \cap \Gamma(g, \Delta_{G_N^{(e)}}) = \emptyset$" instead of "$W_1 \cap \overline{\Gamma(g, \Delta_{G_N^{(e)}})}$ is a polar set", then we have $w \in R(g, \Delta_e)$ for all w: $0 < |w| < 1$ as in the proof of Lemma 1.

If $w_0 \in W_{r/2}$ $(w_0 \neq 0)$, we have $w_0 \in C(g, \Delta_e)$ and $W'_{r/2} \cap \Gamma(g, \Delta_{G_N^{(e)}}) = \emptyset$ $(W'_{r/2} = \{w;\ |w - w_0| < r/2\})$, and hence $0 \in R(g, \Delta_e)$.

Thus we have $W_1 \subset R(g, \Delta_e)$ and $\alpha \in \operatorname{int} R(f, \Delta_e)$.

Theorem 2. $\overline{R(f, \Delta_e)^c} \cap C(f, \Delta_e) \cap R' \subset \chi^*(f, \Delta_e)$.

Proof. From Lemma 3, we have

$$\chi^*(f, \Delta_e)^c \subset C(f, \Delta_e)^c \cup R'^c \cup (\operatorname{int} R(f, \Delta_e));$$

that is,

$$\overline{R(f, \Delta_e)^c} \cap C(f, \Delta_e) \cap R' \subset \chi^*(f, \Delta_e).$$

Lemma 4. $\operatorname{int} C(f, \Delta_e) \subset \overline{R(f, \Delta_e)}$.

Proof. If $\alpha \in \operatorname{int} C(f, \Delta_e)$, for any neighborhood U of α, there exists a parametric disk V_0 about α_0 satisfying $\overline{V_0} \subset U \cap C(f, \Delta_e)$. Since $\alpha_0 \in C(f, \Delta_e)$, there exists $p_1 \in G_1^{(e)}$ such that $\alpha_1 = f(p_1) \in V_0$. We can take a parametric disk V_1 about α_1 satisfying $\overline{V_1} \subset V_0 \cap f(G_1^{(e)})$. Repeating the same method, we have a sequence of parametric disks $\{V_n\}$ $(n = 1, 2, 3, \ldots)$ such that $\overline{V_{n+1}} \subset V_n$ and $\overline{V_n} \subset f(G_n^{(e)})$. $\beta \in \bigcap_n \overline{V_n}$ is assumed by f in any $G_n^{(e)}$, and hence $\alpha \in \overline{R(f, \Delta_e)}$.

Corollary 2. $R(f, \Delta_e)^c \cap R' \subset \chi^*(f, \Delta_e)$ *if and only if* $\overline{R(f, \Delta_e)} = R'^*$.

Proof. If $C(f, \Delta_e) \neq R'^*$, there exists α_0 such that $\alpha_0 \in C(f, \Delta_e)^c \cap R' \subset R(f, \Delta_e)^c \cap R'$. If $\alpha \in \chi^*(f, \Delta_e)$, then we have $\alpha \in \overline{\Gamma(f, \Delta_{G_n^{(e)}})}$ for any n and $0 \in \overline{\Gamma(g, \Delta_{G_n^{(e)}})}$ for a parametric disk V about α. Since there exists $w_n \in W_{1/n} \cap \Gamma(g, \Delta_{G_n^{(e)}})$, there exists $p_n \in G_n^{(e)}$ such that $g(p_n) \in W_{1/n}$. Since p_n converges to e and $g(p_n)$ converges to 0, we see that $0 \in C(g, \Delta_e)$ and $\alpha \in C(f, \Delta_e)$. Hence we have $\chi^*(f, \Delta_e) \subset C(f, \Delta_e)$ and $\alpha_0 \notin \chi^*(f, \Delta_e)$. Thus if $R(f, \Delta_e)^c \cap R' \subset \chi^*(f, \Delta_e)$, from Lemma 4, $\overline{R(f, \Delta_e)} = R'^*$.

Conversely if $\overline{R(f, \Delta_e)} = R'^*$, then we have, from Theorem 2,

$$R(f, \Delta_e)^c \cap R' = R(f, \Delta_e)^c \cap C(f, \Delta_e) \cap R' \subset \chi^*(f, \Delta_e).$$

Corollary 3. *If the characteristic function of f (cf. [3]) is unbounded, then* $R(f, \Delta_e)^c \cap R' \subset \chi^*(f, \Delta_e)$.

Proof. If $C(f, \Delta_e) \neq R'^*$, since f_G is a Lindelöf mapping, as in the proof of Theorem 1, the characteristic function of f is bounded, and a contradiction. Thus from Lemma 4 and Corollary 2 we get $R(f, \Delta_e)^c \cap R' \subset \chi^*(f, \Delta_e)$.

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DEPARTMENT OF MATHEMATICS, SHIBAURA INSTITUTE OF TECHNOLOGY, 3-9-14, SHIBAURA, MINATO-KU, TOKYO, JAPAN