PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 53, Number 1, November 1975

TWO EXAMPLES IN PROXIMITY SPACES

P. L. SHARMA

ABSTRACT. Two examples of Lo-spaces are given. The first is an example of a Lo-space in which not every ultrafilter is contained in a cluster. In the Lo-space of the second example, each ultrafilter is contained in a cluster, and yet not every maximal bunch is a cluster.

It is well known that in Efremovič proximity spaces each maximal bunch is a cluster and also each ultrafilter is contained in a unique cluster. In Example 1 we construct a Lo-space in which not every ultrafilter is contained in a cluster and, consequently, in that space, not every maximal bunch is a cluster. Surprisingly there also exist Lo-spaces in which every ultrafilter is contained in a cluster and still there are maximal bunches which are not clusters. One such space is outlined in Example 2.

We shall be using the terminology of [1], some of which is given below.

Let δ be a binary relation on the power set of a nonempty set X. Consider the following axioms:

(P_0) $(\{x\}, \{y\}) \in \delta$ implies $x = y$;

(P_1) $(\phi, x) \notin \delta$;

(P_2) $(A, B) \in \delta$ implies $(B, A) \in \delta$;

(P_3) $A \cap B \neq \varnothing$ implies $(A, B) \in \delta$;

(P_4) $(A, B \cup C) \in \delta$ if and only if $(A, B) \in \delta$ or $(A, C) \in \delta$;

(P_5) $(A, B) \in \delta$ and $(\{b\}, C) \in \delta$ for each $b \in B$ implies $(A, C) \in \delta$;

(P_6) $(A, B) \notin \delta$ implies that there exists a subset E of X such that $(A, E) \notin \delta$ and $(X - E, B) \notin \delta$.

(i) δ satisfying (P_1–P_5) is called a Lo-proximity.

(ii) δ satisfying (P_1–P_4) and (P_6) is called an Efremovič proximity (or EF proximity).

(iii) δ satisfying (P_0) is called separated.

Clearly every EF proximity is a Lo-proximity but not conversely. If δ is a Lo-proximity (EF proximity) on X, then the pair (X, δ) is called a Lo-space (resp. EF space).

A topological space X is R_0 if and only if for each $x \in X$ and each neighborhood G of X, we have $\overline{\{x\}} \subset G$. A Lo-proximity δ on a set x induces an R_0 topology on X via the Kuratowski closure operator given by $\bar{A} = \{x \in X: (\{x\}, A) \in \delta\}$.

Received by the editors September 30, 1974.
AMS (MOS) subject classifications (1970). Primary 54E05.
Key words and phrases. Lo-proximity, bunch, cluster.

Let (X, δ) be a Lo-space. A collection σ of subsets of X is called a bunch provided σ is nonempty and satisfies the following conditions:

(a) $A, B \in \sigma$ implies $(A, B) \in \delta$;

(b) $A \cup B \in \sigma$ if and only if $A \in \sigma$ or $B \in \sigma$;

(c) $\bar{A} \in \sigma$ implies $A \in \sigma$.

σ is called a cluster if it is a bunch and satisfies the following:

(d) If $G \subset X$ and $(G, A) \in \delta$ for each $A \in \sigma$ then $G \in \sigma$.

A full account of this and other related material is given in [1].

Example 1. This is an example of a Lo-space in which some ultrafilters are contained in no cluster. Since each ultrafilter is contained in a maximal bunch, the Lo-space constructed here will contain maximal bunches which are not clusters.

Let $\mathfrak{F}_1, \mathfrak{F}_2, \mathfrak{F}_3$ and $\mathfrak{F}_4$ be four distinct nonprincipal ultrafilters on an infinite set X, and let $\mathfrak{F}_5 = \mathfrak{F}_1$. Define a binary relation δ on the power set of X as follows:

$A \delta B$ if and only if at least one of the following two conditions is satisfied:

(i) $A \cap B \neq \emptyset$;

(ii) For some i, $1 \leq i \leq 4$, one of the sets A, B is in $\mathfrak{F}_i$ and the other belongs to $\mathfrak{F}_{i+1}$.

It is easy to verify that δ is a Lo-proximity. We claim that the filter $\mathfrak{F}_i$ cannot be contained in any cluster for any i, $1 \leq i \leq 4$. We prove this for the filter $\mathfrak{F}_1$. If possible suppose there exists a cluster σ such that $\mathfrak{F}_1 \subseteq \sigma$. Then $\mathfrak{F}_1 \subseteq \sigma \subseteq \mathfrak{F}_1 \cup \mathfrak{F}_2 \cup \mathfrak{F}_4$. Take any $B \in \mathfrak{F}_2$ and $C \in \mathfrak{F}_3$ such that $C \notin \mathfrak{F}_1 \cup \mathfrak{F}_2 \cup \mathfrak{F}_4$. Then $B \cup C \in \mathfrak{F}_2 \cap \mathfrak{F}_3$. Therefore $(B \cup C) \delta P$ for each $P \in \mathfrak{F}_1 \cup \mathfrak{F}_2 \cup \mathfrak{F}_2$. But since $\sigma \subseteq \mathfrak{F}_1 \cup \mathfrak{F}_2 \cup \mathfrak{F}_4$, $(B \cup C) \delta P$ for each $P \in \sigma$. As σ is a cluster, we must have $B \cup C \in \sigma$. But as $C \notin \mathfrak{F}_1 \cup \mathfrak{F}_2 \cup \mathfrak{F}_4$, there exists $A \in \mathfrak{F}_1$ such that $C \not\delta A$. Consequently $C \notin \sigma$, so $B \in \sigma$. Since $B \in \mathfrak{F}_2$ was arbitrary, we have $\mathfrak{F}_2 \subseteq \sigma$. Similarly $\mathfrak{F}_4 \subseteq \sigma$, and thus $\sigma = \mathfrak{F}_1 \cup \mathfrak{F}_2 \cup \mathfrak{F}_4$. Now take two sets B and D such that $B \cap D = \emptyset$, $B \in \mathfrak{F}_2$, $D \in \mathfrak{F}_4$, $B \notin \mathfrak{F}_1 \cup \mathfrak{F}_3 \cup \mathfrak{F}_4$ and $D \notin \mathfrak{F}_1 \cup \mathfrak{F}_2 \cup \mathfrak{F}_3$. By our choice $B \not\delta D$ and both B and D belong to σ. This is a contradiction. Thus we conclude that there is no cluster containing the filter $\mathfrak{F}_1$. The same is true for the filters $\mathfrak{F}_2$, $\mathfrak{F}_3$ and $\mathfrak{F}_4$.

Example 2. This is an example of a Lo-space, in which, even though each ultrafilter is contained in a cluster, not every maximal bunch is a cluster.

Define a binary relation δ on the power set of the set R of real numbers as follows:

$(A, B) \in \delta$ if and only if at least one of the following four conditions is satisfied.

(i) $A \cap B \neq \emptyset$.

(ii) One of A and B contains an infinite subset of positive integers and the other contains an uncountable subset of positive real numbers.

(iii) One of A and B contains an infinite subset of negative integers and the other contains an uncountable subset of negative real numbers.

(iv) A and B are both uncountable.

The verification of δ being a separated Lo-proximity is straightforward. Also the collection ζ of all uncountable subsets of R can easily be seen to be a bunch. We claim that ζ is a maximal bunch but not a cluster.

To show that ζ is a maximal bunch, take any bunch ζ_1 such that $\zeta \subseteq \zeta_1$. It suffices to show that $\zeta = \zeta_1$. To see this let $A \in \zeta_1$. Write $A^+ = \{n \in A: n \text{ is a positive integer}\}$, $A^- = \{n \in A: n \text{ is a negative integer}\}$ and $B = \{x \in A: x \notin A^+ \cup A^-\}$. Since the set R^- of all negative real numbers is in ζ and $(A^+, R^-) \notin \delta$, then $A^+ \notin \zeta$. Similarly $A^- \notin \zeta$, and therefore $A^+ \notin \zeta_1$ and $A^- \notin \zeta_1$. Since $A \in \zeta_1$ and $A = A^+ \cup A^- \cup B$, then $B \in \zeta_1$. Let E be any uncountable subset of R. Then $E \in \zeta_1$ and therefore $(E, B) \in \delta$. Since B contains no positive integer nor any negative integers and $(E, B) \in \delta$ for any arbitrary uncountable subset E of R, it follows from the definition of δ that B is uncountable and, consequently, so is A. It follows that $A \in \zeta$ and, therefore, $\zeta = \zeta_1$. This proves that ζ is a maximal bunch. To show that ζ is not a cluster it is enough to observe that for the set I of all integers we have $(I, A) \in \delta$ for each $A \in \zeta$, whereas $I \notin \zeta$.

Now we show that each ultrafilter on (R, δ) is contained in a cluster. Take any nonprincipal ultrafilter $\mathfrak{F}$ on R. Then one of the sets $P = \{x \in R: x > 0\}$ and $N = \{x \in R: x < 0\}$ is in $\mathfrak{F}$. Without any loss of generality, assume $P \in \mathfrak{F}$. Let I^+ be the set of all positive integers. At least one of the following three cases holds.

Case I. $I^+ \in \mathfrak{F}$. In this case the collection $\sigma = \{A \subseteq R: A \in \mathfrak{F}$ or A contains an uncountable subset of $P\}$ is a cluster containing $\mathfrak{F}$.

Case II. $I^+ \notin \mathfrak{F}$ and some member of $\mathfrak{F}$ is countable. In this case $\mathfrak{F}$ itself is a cluster.

Case III. Each member of $\mathfrak{F}$ is uncountable. Let $\mathfrak{G}$ be a nonprincipal ultrafilter on I^+. Then $\sigma = \{A \subset R: A \cap P$ is uncountable or A contains some member of $\mathfrak{G}\}$ is a cluster containing $\mathfrak{F}$.

Thus in all cases, $\mathfrak{F}$ is contained in a cluster.

The author wishes to thank the referee for his valuable suggestions.

REFERENCE

1. S. A. Naimpally and B. D. Warrack, *Proximity spaces*, Cambridge Tracts in Math. and Math. Phys., no. 59, Cambridge Univ. Press, London and New York, 1970. MR 43 #3992.

DEPARTMENT OF MATHEMATICS, SOUTHERN ILLINOIS UNIVERSITY, CARBONDALE, ILLINOIS 62901

Current address: Department of Mathematics, University of Kansas, Lawrence, Kansas 66045