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COMPLETE DOMAINS WITH RESPECT TO THE CARATHÉODORY DISTANCE. II

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ABSTRACT. In [1] we have obtained the following result: Let D be a bounded domain in $\mathbf{C}^n$. Suppose there is a compact subset K of D such that for every $x \in D$ there is an analytic automorphism $f \in \operatorname{Aut}(D)$ and a point $a \in K$ such that $f(x) = a$. Then D is a domain of bounded holomorphy, in the sense that D is the maximal domain on which every bounded holomorphic function on D can be continued holomorphically (cf. Narasimhan [2, Proposition 7, p. 127]). Here we shall give a stronger result: Under the same assumptions, D is c-complete. We note that a c-complete domain is a domain of bounded holomorphy, in particular, a domain of holomorphy. A domain of bounded holomorphy, however, need not be c-complete.

Let D be a bounded domain in $\mathbf{C}^n$. Let $p \in D$ and $q \in \overline{D}$. We define $\tilde{c}(p, q) = \lim_{y \in D} \inf_{y \to q} c(p, y)$, where c is the Carathéodory distance on D. A boundary point q of D is called an infinite distance boundary point if there is at least one sequence (q_n) of points of D which converges to q such that $c(p, q_n) \to \infty$ as $n \to \infty$, $p \in D$. This point q is called a stable infinite distance boundary point if $c(p, q_n) \to \infty$ as $n \to \infty$ for every sequence $(q_n) \to q$. We define the minimal boundary distance from p to the boundary of D by $\inf_{q \in \partial D} \tilde{c}(p, q)$. If p is replaced by a compact subset K of D, then the minimal boundary distance from K to the boundary of D is given by $\min_{p \in K}(\inf_{q \in \partial D} \tilde{c}(p, q))$. We denote this by $\min \tilde{c}(K, \partial D)$. We observe that if D has exclusively stable infinite distance boundary points, $\min \tilde{c}(K, \partial D) = \infty$ for every compact subset K of D. If D has an unstable infinite distance boundary point q or a finite distance boundary point q, then $\min \tilde{c}(K, q) < \infty$ for every compact subset K of D.

Theorem. *Let D be a bounded domain in $\mathbf{C}^n$. Suppose there is a compact subset K of D such that for any $x \in D$ there is an analytic automorphism $f \in \operatorname{Aut}(D)$ and a point $a \in K$ such that $f(x) = a$. Then D is c-complete.*

Proof. Assume that D is not c-complete. Then there is a boundary point which is not of stably infinite distance. Let $r = \min \tilde{c}(K, \partial D)$, where K is a compact subset of D in the hypothesis. Fix $q \in \partial D$ such that $\tilde{c}(K, q) = r$. Choose a sequence of points $\{x_n\}$ of D such that $\{x_n\} \to q$

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and $c(x_0, x_n) < r/3$ for all n. Let $f_n \in \mathrm{Aut}(D)$ such that $f_n(x_n) = a_n \in K$ for all n. Since K is compact, $\{a_n\} \to a \in K$. The family $\{f_n\}$ of automorphisms of D is uniformly bounded so that there is a subsequence $\{f_k\}$ which converges uniformly on compact subsets of D to a holomorphic mapping $f\colon D \to \overline{D}$. Then we have $f_k(x_0) \to f(x_0)$ and

$$r/3 \geq c(x_0, x_k) = c(f_k(x_0), f_k(x_k)) = c(f_k(x_0), a_k) \quad \text{for all } k.$$

Since the distance c is continuous, $c(f_k(x_0), a_k) \to c(f(x_0), a)$. Since $a \in K$ and $f(x_0) \in \{x \in \overline{D};\ \min \widetilde{c}(K, x) \leq r/3\} \subset D$, $f(x_0) \in D$. So $f(D) \not\subset \partial D$. By a theorem of Cartan (see, for instance, Narasimhan [2, Theorem 4, p. 78]), f is an automorphism of D. But this is absurd since $f_k^{-1}(a_k) = x_k$, if a is a limit point of $\{a_k\}$ in K, $f^{-1}(a) \in D$. But $\{x_k\}$ has no limit point in D. Hence D is c-complete.

Corollary. *If Γ is a discrete subgroup of* $\mathrm{Aut}(D)$ *such that D/Γ is compact, then D is c-complete.*

Corollary. *If D is a bounded homogeneous domain then D is c-complete.*

Remark. We may also claim the last corollary by the following facts. Since every bounded homogeneous domain in $\mathbf{C}^n$ is biholomorphic to an affinely homogeneous Siegel domain of second kind, and a Siegel domain of second kind is c-complete, a bounded homogeneous domain is c-complete.

REFERENCES

1. D. Kim, *Complete domains with respect to the Carathéodory distance*, Proc. Amer. Math. Soc. 49 (1975), 169–174.

2. R. Narasimhan, *Several complex variables*, Chicago Lectures in Mathematics, 1971.

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