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TORUS-LIKE PRODUCTS OF λ CONNECTED CONTINUA

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ABSTRACT. Recently the author [5] proved that λ connected continua X and Y are arc-like if and only if the topological product $X \times Y$ is disk-like. Here we present an analogous theorem that generalizes the result of Fort [2] and Ganea [3] that disks are not torus-like. We prove that λ connected continua X and Y are circle-like if and only if $X \times Y$ is torus-like.

We call a nondegenerate compact connected metric space a *continuum.* A *map* is a continuous single-valued function.

A continuum X is circle-like if for each positive number ϵ, there is an ϵ-map (i.e., a map such that each point-preimage has diameter $<\epsilon$) of X onto a circle. *Torus-like* continua are defined in the same manner. Here a torus is the cartesian product of two circles.

A continuum is *decomposable* if it is the union of two proper subcontinua. A continuum is *hereditarily decomposable* if all of its subcontinua are decomposable. If each two points of a continuum X can be joined by a hereditarily decomposable subcontinuum of X, then X is said to be λ *connected.*

A continuum Y is called a *triod* if it contains a subcontinuum Z such that $Y - Z$ is the union of three nonempty disjoint open sets. When a continuum does not contain a triod, it is said to be *atriodic.*

A continuum is unicoherent provided that if it is the union of two subcontinua E and F, then $E \cap F$ is connected.

For any two metric spaces (X, ψ) and (Y, ϕ), we shall always assume that the distance between two points $p_1 = (x_1, y_1)$ and $p_2 = (x_2, y_2)$ of the topological product $X \times Y$ is defined by

$$\rho(p_1, p_2) = ((\psi(x_1, x_2))^2 + (\phi(y_1, y_2))^2)^{1/2}.$$

Theorem 1. *Suppose that X and Y are λ connected continua and that $X \times Y$ is torus-like. Then X is atriodic, every proper subcontinuum of X is unicoherent, and X is not unicoherent.*

Proof. Let ψ and ϕ be distance functions for X and Y respectively.

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Define Y_1 and Y_2 to be disjoint subcontinua of Y. Note that if $\epsilon = \phi(Y_1, Y_2)$ and f is an ϵ-map of $X \times Y$ onto a torus, then either $f[X \times Y_1]$ or $f[X \times Y_2]$ can be embedded in a 2-sphere [8, Lemma 1]. It follows from paragraphs 2 through 4 in the proof of Theorem 1 in [5] that X is atriodic. By the argument presented in paragraphs 5 through 13 in the same proof, every proper subcontinuum of X is unicoherent. Note that Y is atriodic and every proper subcontinuum of Y is unicoherent.

Now suppose that X is unicoherent. By Theorem 2 of [5], X is hereditarily decomposable. Hence there is a monotone map g of X onto the unit interval $[0, 1]$ [1, Theorem 8]. Define ϵ_1 to be the minimum of

$$\{\psi(g^{-1}[[0, n/9]], g^{-1}[[(n + 1)/9, 1]]) \mid n = 1, 2, \ldots, 7\}.$$

Assume that Y is unicoherent. Then Y is hereditarily decomposable and there exists a monotone map h of Y onto $[0, 1]$.

Define ϵ to be a positive number less than ϵ_1, $\phi(h^{-1}(0), h^{-1}[[1/3, 1]])$, $\phi(h^{-1}(1), h^{-1}[[0, 2/3]])$, and $\phi(h^{-1}[[0, 1/3]], h^{-1}[[2/3, 1]])$. Let f be an ϵ-map of $X \times Y$ onto a torus T.

At least one of the disjoint continua $f[g^{-1}[[0, 4/9]] \times Y]$ and $f[g^{-1}[[5/9, 1]] \times Y]$ is lying in a planar connected open subset of T. We assume without loss of generality that a planar connected open set S in T contains $f[g^{-1}[[0, 4/9]] \times Y]$.

The continuum $K = f[g^{-1}[[2/9, 1/3]] \times Y]$ separates $L = f[g^{-1}(0) \times Y]$ from $M = f[g^{-1}(4/9) \times Y]$ in T. Hence K separates L from M in S. Note that the intersection of

$$f[g^{-1}[[2/9, 1/3]] \times h^{-1}[[0, 2/3]]] \quad \text{and} \quad f[g^{-1}[[2/9, 1/3]] \times h^{-1}[[1/3, 1]]]$$

is the continuum $f[g^{-1}[[2/9, 1/3]] \times h^{-1}[[1/3, 2/3]]]$. It follows from Janiszewski's theorem [7, Theorem 20, p. 173] that either

$$E = f[g^{-1}[[2/9, 1/3]] \times h^{-1}[[0, 2/3]]]$$

or $f[g^{-1}[[2/9, 1/3]] \times h^{-1}[[1/3, 1]]]$ separates L from M in S.

We assume without loss of generality that E separates L from M in S. But $f[g^{-1}[[0, 4/9]] \times h^{-1}(1)]$ is a continuum in S that meets both L and M and misses E, a contradiction. Hence Y is not unicoherent.

According to Lemma 2 of [6], Y is not separated by any of its subcontinua. By Theorem 5 of [4], there exists a monotone map k of Y onto a circle C.

Define Z_1, Z_2, Z_3, and Z_4 to be arcs whose interiors are pairwise disjoint such that $C = \bigcup_{i=1}^4 Z_i$ and $Z_1 \cap Z_3 = \emptyset = Z_2 \cap Z_4$. Let V_1, V_2, V_3, and V_4 be arcs in C such that $V_1 \cap V_3 = \emptyset = V_2 \cap V_4$, and for each integer i $(1 \leq i \leq 4)$, the interior of V_i contains Z_i.

Define ϵ' to be a positive number less than ϵ_1, $\phi(k^{-1}[V_1], k^{-1}[V_3])$, $\phi(k^{-1}[V_2], k^{-1}[V_4])$, and the minimum of $\{\phi(k^{-1}[Z_i], k^{-1}[C - V_i]) \mid i = 1, 2, 3, \text{ and } 4\}$. Let t be an ϵ'-map of $X \times Y$ onto the torus T.

For each integer i $(1 \le i \le 4)$ define $A_i = t[X \times k^{-1}[V_i]]$. Note that $T = \bigcup_{i=1}^{4} A_i$ and $A_1 \cap A_3 = \varnothing = A_2 \cap A_4$.

Using arcs in T that approximate each $t[g^{-1}(0) \times k^{-1}[Z_i]]$, we define for each i $(1 \le i \le 4)$ an arc α_i in $A_i \cap t[g^{-1}[[0, 2/9]] \times Y]$ such that $\alpha = \bigcup_{i=1}^{4} \alpha_i$ is a simple closed curve. By Fort's lemma [2], there is a retraction r of T onto α. The torus T is not separated by α; for otherwise, r restricted to the closure of the planar component of $T - \alpha$ would be a retraction of a disk onto its boundary, which is impossible. Note that α lies in $t[g^{-1}[[0, 2/9]] \times Y]$.

In a similar manner, we define simple closed curves β and γ contained in $B = t[g^{-1}[[1/3, 2/3]] \times Y]$ and $t[g^{-1}[[7/9, 1]] \times Y]$, respectively, such that neither β nor γ separates T.

Since α, β, and γ are pairwise disjoint, $T - (\alpha \cup \beta \cup \gamma)$ has three components. Let H be the component of $T - (\alpha \cup \beta \cup \gamma)$ whose boundary is $\alpha \cup \gamma$. Note that H does not meet β.

Since B is a continuum in T that contains β and misses $\alpha \cup \gamma$, B does not intersect H. Thus H is contained in the union of disjoint continua

$$A = t[g^{-1}[[0, 1/3]] \times Y] \quad \text{and} \quad G = t[g^{-1}[[2/3, 1]] \times Y].$$

Since α and γ lie in A and G, respectively, it follows that H meets both A and G. But this implies that H is not connected, a contradiction. Hence X is not unicoherent.

Theorem 2. *Suppose that X and Y are λ connected continua. Then X and Y are circle-like if and only if $X \times Y$ is torus-like.*

Proof. If $X \times Y$ is torus-like, then X and Y are both atriodic nonunicoherent λ connected continua with the property that every proper subcontinuum is unicoherent (Theorem 1). It follows from Theorem 2 of [6] that X and Y are circle-like.

To see that the torus-like product condition is also necessary, note that if f is an $\epsilon/2$-map of X onto a circle C and g is an $\epsilon/2$-map of Y onto C, then the function h of $X \times Y$ onto the torus $C \times C$ defined by $h((x, y)) = (f(x), g(y))$ is an ϵ-map.

Question. Must continua X and Y (not necessarily λ connected) be circle-like when $X \times Y$ is torus-like?

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