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THE DOUBLE CENTRALIZER ALGEBRA AS A LINEAR SPACE

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ABSTRACT. Let A be a C^*-algebra and $M(A)$ be the double centralizer algebra of A. The properties semireflexivity, nuclearity, (DF), and strict compactness of the unit ball are characterized in $M(A)$ endowed with the strict topology.

1. **Introduction.** Throughout this paper A denotes a C^*-algebra. A double centralizer of A is a pair (T, S) where T and S are bounded linear operators on A which satisfy the relation $xT(y) = S(x)y$, $\forall\ x, y \in A$. The set of all such pairs, denoted $M(A)$, has a natural C^*-algebra structure and A may be regarded as a closed two-sided ideal of $M(A)$. R. C. Busby [2] initiated the study of $M(A)$ for A a C^*-algebra wherein he defined the strict topology β (generalizing Buck's topology [1]) as that having the seminorms $x \longrightarrow \|xa\|$ and $x \longrightarrow \|ax\|$ for $x \in M(A)$ and $a \in A$. D. C. Taylor [12], [13] has made an extensive study of $M(A)$.

In this paper S denotes a locally compact Hausdorff space and $C_0(S)$ the continuous complex functions vanishing at infinity on S. $C^*(S)$ $(= M(C_0(S)))$ denotes the continuous bounded complex functions on S. If E is a topological vector space, E' denotes its dual. $M(A)_\beta$ denotes $M(A)$ with the strict topology. If no other topology on $M(A)$ is mentioned, we mean the norm topology.

In this paper we characterize nuclearity, semireflexivity, property (DF) and compactness of the unit ball of $M(A)_\beta$.

2. **Results.**

2.1 **Notation.** Let $f \in M(A)'$ and $b \in M(A)$. The functionals $b \cdot f$ and $f \cdot b$ on $M(A)$ are defined by the equations $b \cdot f(x) = f(xb)$ and $f \cdot b(x) = f(bx)$, for $x \in M(A)$. If $b \in A$, then $b \cdot f$ and $f \cdot b \in M(A)'_\beta$ [13, 2.2 and 2.3].

2.2 **Definition.** Let $\{A_p | p \in P\}$ be a family of Banach spaces or C^*-algebras. By ΣA_p we mean $\{x = (x_p) \text{ in } \Pi A_p | \sup_p \|x_p\| < \infty\}$ with pointwise operations. By the subdirect sum or restricted product of the family, denoted

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$(\Sigma A_p)_0$, we mean $\{x = (x_p) \in \Sigma A_p \mid \forall\ \epsilon > 0,\ \|x_p\| > \epsilon$ for at most finitely many $p \in P\}$. By $(\Sigma A_p)_1$, we mean $\{x \in \Sigma A_p \mid \Sigma \|x_p\| < \infty\}$. If $A_p = A$, for all p, ΣA_p is denoted $l^\infty(A)$.

Our first two results generalize results of Collins [3, Theorems 4.1, 4.10].

2.3. **Proposition.** *$M(A)$ has a β-compact unit ball iff A is a subdirect sum (restricted product) of finite-dimensional C^*-algebras.*

Proof. First we show that left or right multiplication by an element of A is a compact operator on A if $M(A)$ has a β-compact unit ball. Let $a \in A$ and B be the unit ball in A. Then B is β-totally bounded so aB and Ba are norm totally bounded subsets of A. Apply [6, 4.7.21].

The converse is clear.

2.4 **Theorem.** *$M(A)_\beta$ is nuclear [9] iff A is finite-dimensional.*

Proof. If $M(A)_\beta$ is nuclear, the unit ball is A is β-compact. By 2.3, $A = (\Sigma A_p)_0$, where each A_p is finite-dimensional; hence $M(A) = \Sigma A_p$ (see 2.2 and [13, 3.1]). In a nuclear space, every unconditionally summable series is absolutely summable [9, p. 184]. Suppose that the index set P is infinite. Choose an infinite sequence $\{p_n\}_{n=1}^\infty$ from P, let e_n denote the identity in A_{p_n} and let e'_n be the element of $M(A)$ which has value e_n in the p_nth coordinate and zero in the other coordinates. Since $\{e'_n\}_{n=1}^\infty$ is unconditionally summable, but not absolutely summable, P is finite; hence A is finite-dimensional.

The converse is clear.

Our next result concerns property (DF) of Grothendieck and a related property (WDF) defined by Collins and Dorroh (see [4, pp. 162–163] for definitions). In [4] it is shown that (DF) implies (WDF) and in [11] Summers gives the converse. We extend these results in 2.7.

2.5 **Definition.** Let A be a Banach algebra with approximate identity and V a Banach space which is an A-module with action of A on V denoted by $a \cdot v$ for $a \in A$, $v \in V$. V is called *essential* if for some (or, for all) approximate identity $\{e_p \mid p \in P\}$ for A, $\|e_p \cdot v - v\| \xrightarrow[p]{} 0$ for each $v \in V$.

2.6 **Definition.** Let A be a C^*-algebra. $M(A)_\beta$ is (WDF) if, for each Hermitian sequence $\{a_n\}_{n=1}^\infty$ in A, $V = \bigcap_{n=1}^\infty V_n$ absorbs points of $M(A)$ implies V is a β zero neighborhood in $M(A)$ where $V_n = \{x \in M(A) \colon \|xa_n\| \le 1$ and $\|a_n x\| \le 1\}$.

2.7 **Theorem.** *The following implications hold*: (1) *iff* (2) *iff* (3) *and* (3) *implies* (4).

(1) $M(A)_\beta$ *is* (DF);

(2) $M(A)_\beta$ *is* (WDF);

(3) *$l^\infty(A)$ is an essential A-module under both left and right actions*;

(4) *If A is separable, then A has an identity.*

Proof. (1) implies (2) is clear. To show (2) $\to$ (3), let $\{e_p | p \in P\}$ be an approximate identity for A and let $a = \{a_n\} \in l^\infty(A)$. Let $V = \bigcap_{n=1}^\infty V_n$ where $V_n = \{x \in M(A): \|xa_n\| \leq 1$ and $\|a_n x\| \leq 1\}$. Since V is closed, absolutely convex, and absorbs points, V is a β zero neighborhood. As $e_p \underset{p}{\to} I$ (the identity of $M(A)$ in the strict topology), $e_p - I \in V$ eventually so $\|(e_p - I)a_n\|$ and $\|a_n(e_p - I)\|$ both converge to zero uniformly in n. Thus $l^\infty(A)$ is essential.

For (3) implies (4), note $e_p x \to x$ uniformly on the unit ball of A since A is separable and $l^\infty(A)$ is an essential A-module. Therefore by [10, Theorem 3.2, p. 146], $e_p \to I$ uniformly, so A has an identity.

For (3) implies (1), let $H = \bigcup_{n=1}^\infty H_n$ be a subset of the unit ball in $M(A)'_\beta$ [13, 2.2] and each H_n be equicontinuous. We must show that H is equicontinuous. Examining the proof of [13, 2.6] we see that $\exists a = \{a_n\}_{n=1}^\infty \in l^\infty(A)$ with $\|a\| < 8$ such that $H_n \subseteq V_n^\circ$ (polar of V_n) where $V_n = \{x \in M(A)$: $\|xa_n\| \leq 1$ and $\|a_n x\| \leq 1\}$. Thus $H \subseteq \bigcup_{n=1}^\infty V_n^\circ$, so [9, p. 126] we have that $V_n^\circ \subseteq$ the β-weak-* closure of the set sum $X_n^\circ + Y_n^\circ$ where $Y_n = \{x \in M(A)$: $\|xa_n\| \leq 1\}$ and $X_n = \{x \in M(A)$: $\|a_n x\| \leq 1\}$. Let β_1 and β_2 denote the topologies on $M(A)$ given by right and left multiplications, respectively, by elements of A. In [13, 2.1–2.2], [10] it is shown that $M(A)'_{\beta_1} = M(A)'_{\beta_2} = M(A)'_\beta$. Thus, by Alaoglu's theorem applied to $M(A)_{\beta_1}$ and $M(A)_{\beta_2}$, X_n° and Y_n° are β-weak-* compact and, hence, $X_n^\circ + Y_n^\circ$ is β-weak-* closed in $M(A)_\beta$. We now calculate X_n° and Y_n°. Clearly $\{f \cdot a_n : f \in M(A)'$ and $\|f\| \leq 1\} \subseteq X_n^\circ$ (see 2.1). On the other hand, if $h \in X_n^\circ$, then $|h(x)| \leq \|a_n x\|$ for all $x \in M(A)$ so the linear functional g defined on $a_n M(A)$ by $g(a_n x) = h(x)$ satisfies $\|g\| \leq 1$. Extend g to f in $M(A)'$ such that $\|f\| \leq 1$. Then, if $x \in M(A)$, $f \cdot a_n(x) = f(a_n x) = g(a_n x) = h(x)$. Thus $f \cdot a_n = h$, so $X_n^\circ = \{f \cdot a_n : \|f\| \leq 1\}$. Similarly, $Y_n^\circ = \{a_n \cdot f : \|f\| \leq 1\}$. Thus $H \subseteq \{f \cdot a_n + a_n \cdot g : \|f\| \leq 1, \|g\| \leq 1$, $n = 1, 2, 3, \ldots\}$. Using (3), let $a \in A$ and $\{b_n\} \in l^\infty(A)$ such that $ab_n = b_n^* a = a_n$. Using Taylor's criterion [13, Theorem 2.6], it is now clear that H is β-equicontinuous. Finally, since β-bounded sets are uniformly bounded [10, Theorem 4.6, p. 149], we see that $M(A)_\beta$ is (DF).

Taylor [13, 2.1–2.3] has shown that $M(A)'_\beta$ "is" A'. Our next result answers the questions of when the resulting natural injection of $M(A)$ into A'' is onto. This result has been obtained also in [8], [14] using different proofs and from points of view different from ours.

2.8 **Lemma.** *Let A be a regular (and Hausdorff) topological space and F a subspace of A. F is relatively compact if every net in F has a cluster point in A.*

2.9 **Theorem.** *$M(A)_\beta$ is semireflexive* [9, p. 143] *iff A is a dual C^*-algebra* [6, p. 99].

Proof. If A is dual, $A = (\Sigma A_p)_0$ (see 2.2) where A_p is the algebra of compact operators on some Hilbert space H_p [6, p. 99]. Since $M(A_p) = B(H_p)$ [2], $M(A) = \Sigma B(H_p)$ [13, 3.1], so $M(A)'_\beta = (\Sigma A'_p)_1$ and $M(A)''_\beta = (\Sigma A'_p)'_1 = \Sigma A''_p = \Sigma B(H_p)$ since the bidual of the algebra of compact operators on H_p is $B(H_p)$ [6]. Since the strong topology on $M(A)'_\beta$ equals the norm topology on $(\Sigma A'_p)_1$, we see that $M(A)_\beta$ is semireflexive.

For the converse, we show that aF and Fa are relatively weakly compact subsets of A, where $a \in A$ and F is the unit ball of A [6, p. 99]. Let x_p be a net in F and $a \in A$. By semireflexivity, the unit ball B in $M(A)$ is β-weakly compact, so x_p clusters to $x_0 \in B$. Let $f \in A'$ and g be a norm-preserving extension of f to $M(A)$. Then $g \cdot a \in M(A)'_\beta$ [13, 2.2], so $\{g \cdot a(x_p - x_0)\}$ clusters to zero. Hence $\{f(ax_p - ax_0)\}$ clusters to zero, i.e., $\{ax_p\}$ clusters weakly to ax_0. Conclude using 2.8.

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