EPIMORPHISMS IN THE CATEGORY OF ABELIAN
l-GROUPS

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ABSTRACT. By choosing the mappings for a category of l-groups care-
fully, some significant categorical results are obtainable. The intent of
this paper is to investigate the reasons why epimorphisms in the category
of l-homomorphisms between abelian l-groups are not "onto" functions.

Introduction. It was observed by D. Topping [5] that the category of all
l-homomorphisms between l-groups contains no injectives. In view of this
result, P. Conrad [2] showed that in a more restrictive category (defined by
N. Popa, 1965) the essentially closed l-groups are the injectives. Thus,
by choosing the category carefully, some significant categorical results are
obtainable. The intent of this paper is to investigate the reasons why epi-
morphisms in the category of l-homomorphisms between abelian l-groups are
not "onto" functions. The definitions for the categorical notions are taken
from Freyd, *Abelian categories.* §1 contains definitions, examples, and some
results concerning epimorphisms. §2 discusses the relationship between epi-
morphisms and archimedean extensions. §3 considers an extension of an l-
group G to another l-group H so that the extended epimorphism to H is an
onto function. As a consequence of this extendability, certain types of l-
groups are shown to have the property that all epimorphisms are onto functions.

1. For clarity of the exposition, the terms l-homomorphisms and l-iso-
morphisms are used when referring to functions from the theory of l-groups;
morphisms, monomorphisms and epimorphisms are used when referring to
functions from the category.

Definition 1. Let $\widetilde{\mathfrak{L}}$ denote the category of all l-homomorphisms between
abelian l-groups.

Definition 2. A *monomorphism* $f \in \widetilde{\mathfrak{L}}$ is a morphism $f: G \to H$ such
that if $g: L \to G$, $h: L \to G$ and $f \circ g = f \circ h$, then $h = g$.

Definition 3. An *epimorphism* $f \in \widetilde{\mathfrak{L}}$ is a morphism $f: G \to H$ such that
if $g: H \to L$, $h: H \to L$ and $h \circ f = g \circ f$, then $h = g$.

Definition 4. An *isomorphism* $f \in \widetilde{\mathfrak{L}}$ is a morphism $f: G \to H$ such that

there exists $g: H \to G$ such that $g \circ f = i_G$.

The reader can easily verify that the classes in question are categories; monomorphisms are one to one l-homomorphisms, and isomorphisms are l-iso-morphisms. If $f \in \bar{\mathfrak{L}}$ and f is a monomorphism which is also an onto l-homo-morphism, then f is an isomorphism. If $f \in \bar{\mathfrak{L}}$, then the first isomorphism theorem is true; i.e., if $f(G)$ denotes the image of G in H, then $G/k(f)$ is l-isomorphic to $f(G)$ [4, Theorem 7, Chapter 2].

The following two examples illustrate that, in general, epimorphisms are not onto l-homomorphisms and that this may or may not depend on the orders involved.

Example 1. This is an example of an epimorphism $f \in \bar{\mathfrak{L}}$ which is not onto as an l-homomorphism. It is also an epimorphism in the category of homomorphisms between torsion free groups. Let $G = 2I$ and $H = I$ where I is the set of integers with their natural order. Let $f: G \to H$ by $f(x) = x$. Consider $g: H \to L$ and $h: H \to L$ such that $g \circ f = h \circ f$. $g(2) = h(2) = 2h(1) = 2g(1)$. Thus, $2(g(1) - h(1)) = 0$. Since L is torsion free, $g(1) = h(1)$, and $g = h$, since the values of $g(1)$ and $h(1)$ determine g and h.

Example 2. This is an example of an epimorphism $f \in \bar{\mathfrak{L}}$ which is not an onto l-homomorphism and is also not an epimorphism in the category of homomorphisms between torsion free groups. Let $G = \{m\sqrt{2}: m \in I\}$ and $H = \{n + m\sqrt{2}: n, m \in I\}$. Use addition and the natural order on both G and H. Define $f: G \to H$ by $f(m\sqrt{2}) = m\sqrt{2}$. Let $g: H \to L$ and $h: H \to L$ such that $g \circ f = h \circ f$. Since H is archimedean, $g(x) = 0$ for some $x > 0$ implies that g is the zero function. Therefore, we may assume that g and h are monomorphisms. Without loss of generality we may also assume that $g(H)$ and $h(H)$ are subgroups of the real numbers. Using [4, Proposition 2, Chapter 4], it follows that g and h are of the form rx for some real number r. Since g and h agree on $\sqrt{2}$, they are equal. To see that f is not an epimorphism in the category of homomorphisms between torsion free groups, define h and g as follows:

$$g(n + m\sqrt{2}) = n, \qquad h(n + m\sqrt{2}) = 0.$$

Note that g does not preserve order.

The following propositions point out a few facts about epimorphisms, some of which will be used in later sections of this paper.

Proposition 1. *If $f \in \bar{\mathfrak{L}}$ and f is an epimorphism from G to H, then the convex hull of $f(G)$ is equal to H.*

Proof. Denote the convex hull by $[f(G)]$. Let α be the natural l-homomorphism from H to $H/[f(G)]$. Let ϕ be the zero function from H to $H/[f(G)]$. $\phi \circ f = \alpha \circ f$ which implies that $\phi = \alpha$ or that $H = [f(G)]$.

Proposition 2. *If f is an epimorphism from G to H, $f \in \bar{\mathfrak{L}}$, and $H =$*

$H_1 \boxplus H_2$, then the composition of f and either projection map is an epimorphism from G to H_i.

Proposition 3. *If $f \in \overline{\mathcal{L}}$ is an epimorphism from G to H and $G = G_1 \boxplus G_2$, then $H = [f(G_1)] \boxplus [f(G_2)]$.*

Proof. Let $0 \leq x \in [f(G_1)] \cap [f(G_2)]$. Since $f(G^+) = f(G)^+$, there exists $0 < z, y$ such that $f(z) \in f(G_1)^+$ and $f(y) \in f(G_2)^+$ and $x \leq f(z), f(y)$. Thus, $f(z) \wedge f(y) \geq x$. Therefore, $f(z \wedge y) \geq x \geq 0$ which implies that $x = 0$. Since $[f(G_1)] \boxplus [f(G_2)]$ is a convex l-subgroup of H which contains $f(G)$, $H = [f(G_1)] \boxplus [f(G_2)]$.

Proposition 4. *If D is the divisible closure of the l-group G and D has the natural order extension from the order on G, then the identity from G to D is an epimorphism.*

Proposition 5. *If f is an epimorphism from G to H, and D and D' are the divisible closures of G and H, respectively, then there exists $g: D \to D'$ such that $g|G = f$.*

Proof. Define $g(x) = f(nx)/n$ where n is the least positive integer such that $nx \in G$. g is well defined since G is torsion free. If $y \in G$, then $g(y) = f(y)$. Let $g(x + y) = f(n(x + y))/n$, $g(x) = f(mx)/m$, and $g(y) = f(ky)/k$. Let $s = \text{l.c.m.}\{n, m, k\}$. Since f is a homomorphism it follows that $sf(n(x + y))/n = f(s(x + y))$, $sf(mx)/m = f(sx)$, $sf(ky)/k = f(sy)$. Thus, $sg(x + y) = sg(x) + sg(y)$ or $s(g(x + y) - g(x) - g(y)) = 0$. Since D' is torsion free, we have that $g(x + y) = g(x) + g(y)$. The order preserving properties of g follow in a similar manner.

2. In order to obtain some connection between the order on H and the l-subgroup $f(G)$ where f is an epimorphism from G to H, we consider the archimedean classes of H.

Definition 5. For two positive elements x and y, x is said to be a-equivalent to y (denoted by $x \sim y$) if there exist two positive integers n and m such that $x \leq ny$ and $y \leq mx$.

Definition 6. An l-subgroup G of H is said to be a-equivalent to H if for all $0 < x \in H$ there exists $0 < y \in G$ such that $x \sim y$.

From a paper by P. Conrad [1], it is noted that if $f(G)$ is an a-equivalent l-subgroup of H, then there exists an l-isomorphism σ from the lattice of all convex l-subgroups of $f(G)$ onto the lattice of all convex l-subgroups of H.

Theorem 1. *If $f: G \to H$ is an epimorphism, $f \in \overline{\mathcal{L}}$, and if f is an epimorphism in the category of homomorphisms between torsion free groups, then H is an a-extension of $f(G)$.*

Proof. Suppose that H is not an a-extension of $f(G)$. Then there exists a positive x in H such that no y in $f(G)$ is a-equivalent to x. Let D denote the divisible hull of H. Consider $f' = i \circ f$ where i is the identity from H to D. f' is an epimorphism in the category of homomorphisms between torsion free groups. Consider D as a vector space over the rationals. Let S_x denote the subspace of D generated by x. Let $\overline{f'(G)}$ denote the subspace of D generated by $f'(G)$. Suppose that $S_x \cap \overline{f'(G)}$ is not the zero subspace and let z be a nonzero element in $S_x \cap \overline{f'(G)}$. $bz = ax$ for a and b integers. $dz = y$ for some $y \in f'(G)$ where d is an integer. Thus, $dax = by$ which contradicts our choice of x. Choose a basis for $\overline{f'(G)}$ and extend it to $\overline{f'(G)} + S_x$ with the addition of x. Then extend this basis for $\overline{f'(G)} + S_x$ to the space D. Let g be the homomorphism which maps x to zero and $y \to y$ for all basis elements $y \neq x$. g will agree with the identity homomorphism on $f'(G)$. This is a contradiction to f' being an epimorphism.

Definition 7. An epimorphism $g: A \to B$ is an a-*epimorphism* if g is also an epimorphism in the category of homomorphisms between torsion free groups.

Note that Theorem 1 implies that for any a-epimorphism $g: A \to B$, $g \in \overline{\mathfrak{L}}$, B is an archimedean extension of $g(A)$.

3. Throughout this section all the morphisms will be from $\overline{\mathfrak{L}}$.

Definition 8. (B, α) is a *completion* of A if $\alpha: A \to B$ is both a monomorphism and an a-epimorphism.

Definition 9. $(\hat{A}, \alpha)$ is an *epicompletion* of A if

(i) $(\hat{A}, \alpha)$ is a completion of A,

(ii) for all a-epimorphisms $f: A \to B$, there exists a completion $(\overline{B}, \delta)$ of B and an "onto" morphism $\hat{f}: \hat{A} \to \overline{B}$ such that the following diagram commutes:

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ \alpha \downarrow & {\scriptstyle \delta} \downarrow & \\ \hat{A} & \xrightarrow{\hat{f}} & \overline{B} \end{array}$$
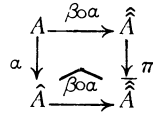
Definition 10. An l-group G is *epicomplete* if all a-epimorphisms $f: G \to H$ are onto functions.

Proposition 6. *If $(\hat{\hat{A}}, \beta)$ is an epicompletion of $(\hat{A}, \alpha)$, then $(\hat{\hat{A}}, \beta \circ \alpha)$ is an epicompletion of A.*

Proof. Since $\hat{f}$ is onto, $\hat{f}$ is an a-epimorphism, and can be extended to $\hat{\hat{f}}$.

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ \alpha \downarrow & {\scriptstyle \delta} \downarrow & \\ \hat{A} & \xrightarrow{\hat{f}} & \overline{B} \\ \beta \downarrow & {\scriptstyle \delta'} \downarrow & \\ \hat{\hat{A}} & \xrightarrow{\hat{\hat{f}}} & \overline{\overline{B}} \end{array}$$
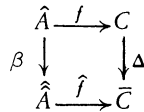
Proposition 7. *In the following diagram,* $\widehat{\beta \circ \alpha} = \pi \circ \beta$.

$$\begin{array}{ccc}
A & \xrightarrow{\beta \circ \alpha} & \hat{A} \\
\alpha \downarrow & & \downarrow \pi \\
\hat{A} & \xrightarrow{\widehat{\beta \circ \alpha}} & \hat{\hat{A}}
\end{array}$$

Proposition 8. $\hat{A} \simeq \hat{\hat{A}}$.

Proof. From the diagram and conclusion of Proposition 7, it follows that π is onto since $\widehat{\beta \circ \alpha}$ is onto. Therefore π is an isomorphism. Suppose there exists $g \in \hat{\hat{A}} \backslash \beta(\hat{A})$. Then there exists $z \in \hat{A}$ such that $\pi \circ \beta(z) = \pi(g)$. Since π is a monomorphism, $\beta(z) = g$. This contradicts our choice of g. Therefore, β is onto and $\hat{A} \simeq \hat{\hat{A}}$.
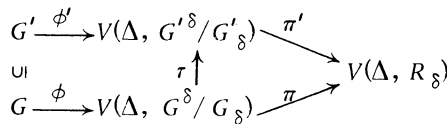
Proposition 9. *If* $(\hat{A}, \alpha)$ *is an epicompletion of A, then any a-epimorphism* $f: \hat{A} \to C$ *is an onto function.*

$$\begin{array}{ccc}
\hat{A} & \xrightarrow{f} & C \\
\beta \downarrow & & \downarrow \Delta \\
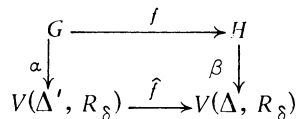\hat{\hat{A}} & \xrightarrow{\hat{f}} & \bar{C}
\end{array}$$

Proof. Let $c \in C$. There exists $g \in \hat{A}$ such that $\hat{f} \circ \beta(g) = \Delta(c)$. Since the diagram commutes, $\Delta \circ f(g) = \Delta(c)$. Δ is a monomorphism; thus, $f(g) = c$.

The previous material presented in this section has not considered the question as to whether $\hat{A}$ will exist for a given l-group A. A partial answer to this question will be stated in Theorem 2. The existence of epicomplete l-groups is established in Corollary 2 to Theorem 2. Needed for the proof of Theorem 2 is a result of P. Conrad's which is given below along with its accompanying diagram.

Theorem 4.1 [1]. *Suppose that G is a divisible abelian l-group, Δ is a plenary subset of $\Gamma_1(G)$ and $V(\Delta, R_\delta) = V$. Then there exists a fixed l-isomorphism ϕ of G into V such that if G' is any divisible a-extension of G, then ϕ can be extended to an l-isomorphism of G' into V.*

$$\begin{array}{ccc}
G' & \xrightarrow{\phi'} V(\Delta, G'^\delta / G'_\delta) \xrightarrow{\pi'} & \\
\cup| & \tau \uparrow & V(\Delta, R_\delta) \\
G & \xrightarrow{\phi} V(\Delta, G^\delta / G_\delta) \xrightarrow{\pi} &
\end{array}$$

Theorem 2. *If $f: G \to H$ is an a-epimorphism, then there exist representations for G and H, $V(\Delta', R_\delta)$ and $V(\Delta, R_\delta)$, respectively (α and β are embedding maps which go with the choice of Δ' and Δ), and an "onto" morphism $\hat{f}$ so that the following diagram commutes:*

$$\begin{array}{ccc}
G & \xrightarrow{f} & H \\
\alpha \downarrow & & \downarrow \beta \\
V(\Delta', R_\delta) & \xrightarrow{\hat{f}} & V(\Delta, R_\delta)
\end{array}$$

Proof. Using Proposition 5, assume that G and H are divisible. Since the construction of the embedding for a divisible abelian l-group into a $V(\Delta, R_\delta)$ is to be used explicitly in this proof, a preliminary outline is given. A plenary subset Δ is chosen from the root system $\Gamma_1(G)$. $\Gamma_1(G)$ is the collection of all regular l-ideals of G. A choice of a Banachewski function π is used to construct the l-isomorphism. The mapping π is a mapping of all subspaces of G into itself such that for all subspaces A and B of G, (i) $G = A \oplus \pi(A)$, and (ii) if $A \subseteq B$, then $\pi(B) \subseteq \pi(A)$. The embedding map ϕ from G to $V(\Delta, G^\delta/G_\delta)$ is defined by

$$\phi(x)_\delta = x_\delta + G_\delta \quad \text{where } x = x_\delta + c_\delta, \ x_\delta \in G^\delta \text{ and } c_\delta \in \pi(G^\delta).$$

The extension from $V(\Delta, G^\delta/G_\delta)$ to $V(\Delta, R_\delta)$ is made by using the existence of an o-isomorphism from G^δ/G_δ to a subgroup of the real numbers.

The proof of Theorem 2 will be reduced in the following way. According to the diagram given with 4.1, let $f(G) = G$ and $H = G'$. Then the following diagram is commutative:

$$
\begin{array}{ccc}
f(G) & \xrightarrow{\ i\ } & H \\
\downarrow{\pi\circ\phi} & & \downarrow{\pi'\circ\phi'} \\
V(\Delta, R_\delta) & \xrightarrow{\ i\ } & V(\Delta, R_\delta)
\end{array}
$$

The existence of $\hat{f}$ now reduces to exhibiting the existence of f', α, Δ', and Δ so as to complete the following diagram:

$$
\begin{array}{ccc}
G & \xrightarrow{\ f\ } & f(G) \\
\downarrow{\alpha} & & \downarrow{\pi\circ\phi} \\
V(\Delta', R_\delta) & \xrightarrow{\ f'\ } & V(\Delta, R_\delta)
\end{array}
$$

The choice of Δ comes later, and then Theorem 4.1 will give $\pi \circ \phi$. So the problem reduces to finding an f' which will be onto.

To further reduce the proof, let γ be the l-isomorphism from G/K to $f(G)$ where K is the kernel of f and $\gamma(x + K) = f(x)$. The composition of the natural homomorphism θ from G to G/K with the function γ will equal f. Therefore, we may consider the following diagram:

$$
\begin{array}{ccccc}
G & \xrightarrow{\ \theta\ } & G/K & \xrightarrow{\ \gamma\ } & f(G) \\
\downarrow{\alpha} & & & & \downarrow{\pi\circ\phi} \\
V(\Delta', R_\delta) & \xrightarrow{\ f''\ } & V(\Delta, R_\delta) & \xrightarrow{\ i\ } & V(\Delta, R_\delta)
\end{array}
$$

Now, with the choice of Δ, $\pi \circ \phi$ will be given by Theorem 4.1, and we need only exhibit an f'' which is onto.

Choose Δ', a plenary set for G, such that $g \notin K$ implies that there exists a value G_δ of g in Δ' such that $G \supseteq K$. Let $\overline{\Delta} = \{G_\delta \in \Delta' : G_\delta \supseteq K\}$. Then, $\Delta = \{G_\delta/K : G_\delta \in \overline{\Delta}\}$ is a plenary set for G/K, and $\overline{\Delta}$ is, by construction, an upper class of Δ'. Moreover, $K = \{g \in G : g$ does not have a value in $\overline{\Delta}\}$.

Let $\alpha'\colon G \to V(\Delta', G^\delta/G_\delta)$ be the embedding function constructed from the Banachewski function, π. Define $\tau'\colon G/K \to V(\Delta, (G^\delta/K)/(G_\delta/K))$ by $\tau'(g + K)_\delta = (g_\delta + K) + G_\delta/K$. Note that τ' is an embedding since

$$G/K = (G^\delta \oplus \pi(G^\delta))/K \simeq G^\delta/K \oplus \pi(G^\delta).$$

Define $g\colon V(\Delta', G^\delta/G_\delta) \to V(\Delta, (G^\delta/K)/(G_\delta/K))$ by $g(v)_\delta = (g_\delta + K) + G_\delta/K$ where $v_\delta = g_\delta + G_\delta$ for each $\delta \in \Delta$. For each coordinate $\delta \in \Delta$, this is the natural mapping of G^δ/G_δ onto $(G^\delta/K)/(G_\delta/K)$. Thus, g is an l-homomorphism, and $\tau' \circ \theta = g \circ \alpha'$.

Let h_δ denote the natural o-isomorphism between G^δ/G_δ and $(G^\delta/K)/(G_\delta/K)$. Let t_1 denote an o-isomorphism between G^δ/G_δ and R_δ. The composition $t_1 \circ h_\delta^{-1}$ is an o-isomorphism from $(G^\delta/K)/(G_\delta/K)$ to R_δ. Let $\tau_2 = h_\delta^{-1} \circ t_1$, and the following diagram commutes:

$$
\begin{array}{ccc}
G^\delta/G_\delta & \xrightarrow{\ h_\delta\ } & (G^\delta/K)/(G_\delta/K) \\
\downarrow{\scriptstyle t_1} & & \downarrow{\scriptstyle t_2} \\
R_\delta & \xrightarrow{\quad i \quad} & R_\delta.
\end{array}
$$

The function f'' is constructed in the obvious manner using the above diagram. Furthermore, f'' is onto. Thus, the existence of $\hat{f}$ has been established.

Corollary 1. *If G is an l-group and Δ is a plenary set for G such that the embedding map π of G into $V(\Delta, R_\delta)$ is an a-epimorphism, then there exists an epicompletion of G.*

Corollary 2. $V(\Delta, R_\delta)$ *is an epicomplete l-group.*

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