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A REFINEMENT OF GREEN'S THEOREM ON THE DEFECT GROUP OF A p-BLOCK

T. Y. LAM[1]

ABSTRACT. Let D be the defect group of a p-block of a finite group G. Let P, Q be two p-Sylow groups of G containing D. Then there exist $x, y, z \in C_G(z)$ such that: (i) z is p-regular and D is a p-Sylow group of $C_G(z)$; (ii) $D = Q^x \cap P = Q \cap P^y$; and (iii) $z = xy$. This refines an earlier theorem of J. A. Green.

Let D be the defect group of a p-block of a finite group G. If P is a p-Sylow group of G containing D, it has been known for some time that there must exist another p-Sylow group Q such that $P \cap Q = D$. This so-called 'Sylow intersection' property was first established by Green [2] using vertex theory, and later also proved by Thompson [4] using Brauer's methods. In [1, p. 241] Alperin indicated that D can even be expressed as a *tame* intersection of two suitable p-Sylow groups of G.

In 1968, a much more accurate result was obtained by Green [3]. If P is any given p-Sylow group of G containing D, Green proved that there must exist $x, y, z \in C_G(D)$ such that (i) z is p-regular and D is a p-Sylow group of $C_G(z)$; (ii) $D = P \cap P^x = P \cap P^y$; (iii) $z = xy$. In particular, if P is chosen such that $N_P(D)$ is a p-Sylow group of $N_G(D)$, then $D = P \cap P^x$ in (ii) clearly expresses D as a tame Sylow intersection. On the other hand, the conclusion of (i) recaptures the earlier result of Brauer that D must be a class-defect group of some p-regular conjugacy class in G.

Green's proof of the above result is a rather elaborate (but extremely successful) application of vertex theory, in the more general framework of G-algebras [3]. For $\mathfrak{O}$ a sufficiently large $\mathfrak{p}$-adic ring ($p|\mathfrak{p}$), Green views the integral group ring $\Gamma = \mathfrak{O}G$ as a (left) $G \times G$-algebra via the action $(g_1, g_2) \cdot \gamma = g_1 \gamma g_2^{-1}$ ($g_i \in G, \gamma \in \Gamma$). Let $E \in \Gamma$ be the $\mathfrak{p}$-adic idempotent associated with the given p-block, whose defect group is D. Given a p-Sylow group $P \supset D$, the following steps are important ingredients in Green's proof:

(1) The vertex of the indecomposable $\mathfrak{O}(G \times G)$-module $\Gamma \cdot E$ is $\Delta(D) = \{(d,d): d \in D\}$.

(2) When viewed as $\mathfrak{O}(P \times P)$-module by restriction, $\Gamma \cdot E$ has at least one indecomposable constituent with vertex $= \Delta(D) \subset P \times P$.

(3) Let $\bigcup_{w \in W} PwP$ be the (P,P)-double coset decomposition of G, and let $[PwP]$ denote the $\mathfrak{O}(P \times P)$-submodule of Γ, with PwP as $\mathfrak{O}$-basis. Then,

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$\Gamma = \oplus_{w \in W}[PwP]$ is a Krull-Schmidt decomposition of Γ in the category of $\mathfrak{O}(P \times P)$-modules. The vertex of $[PwP]$ is $P(w) = \{(p_1, p_2) \in P \times P{:}p_1 w p_2^{-1} = w\}$. Thus, by (2), at least one such $P(w)$ must be conjugate in $P \times P$ to $\Delta(D)$. We may assume $\Delta(D) = P(w_1)$, and let $\{w_1, \ldots, w_c\}$ be a maximal subset of W such that $P(w_j)$ $(1 \leqslant j \leqslant c)$ are mutually nonconjugate in $P \times P$. Since $P(p_1^{-1} w p_2) = P(w)^{(p_1, p_2)}$, we may assume W to have been so chosen that each $P(w)$ $(w \in W)$ is actually *equal* (not just conjugate) to some (unique) $P(w_j)$.

(4) The endomorphism algebra $\mathrm{End}_{P \times P}\,\Gamma$, modulo its radical, can be *explicitly* determined, in the form $\prod_{j=1}^{c} \mathbf{M}_{m_j}(\mathfrak{O}/\mathfrak{p})$, a product of matrix algebras over the modular field $\mathfrak{O}/\mathfrak{p}$. To be precise, the matrices in the jth factor have rows and columns suffixed by $W_j = \{w \in W : P(w) = P(w_j)\}$, and so $m_j = |W_j|$.

(5) By left multiplication, the center of Γ acts as $G \times G$- (hence $P \times P$-) endomorphisms on Γ. Using the projections $\mathrm{End}_{P \times P}\,\Gamma \to \mathbf{M}_{m_j}(\mathfrak{O}/\mathfrak{p})$ in (4), one obtains a family of 'representations' Ψ_j : Center $\Gamma \to \mathbf{M}_{m_j}(\mathfrak{O}/\mathfrak{p})$. If $\mathfrak{K}$ is a typical conjugacy class in G, and K is its class sum in Γ, then $\Psi_j(K)$ will be a matrix whose (u,v)-entry $(u, v \in W_j)$ is the cardinality of the following set (taken mod $\mathfrak{p}$):

$$F_{u,v}^{j}(\mathfrak{K}; P) = \{x \in PuP : P(x) = P(w_j) \text{ and } xv^{-1} \in \mathfrak{K}\}.$$

(6) On the other hand, Ψ_j sends the given $\mathfrak{p}$-adic idempotent E to a matrix $\Psi_j(E)$ whose rank happens to be the number of indecomposable constituents of $(\Gamma \cdot E)_{P \times P}$ with vertex conjugate in $P \times P$ to $P(w_j)$.

The proofs of the above facts vary in degree of difficulty; taken together, they provide an extremely useful tool with which to study the defect group D.

In the following, we shall show how the above information can be further strengthened. Namely, instead of working with *one* p-Sylow group $P \supset D$, one could work *simultaneously* with *two* given p-Sylow groups $P \supset D$, $Q \supset D$. By recasting Green's original methods, we shall show that the following refinement of Green's theorem holds:

THEOREM . *Let D be the defect group of a p-block of the finite group G. Let P, Q be two p-Sylow groups of G containing D. Then there exist $x, y, z \in C_G(D)$ such that*: (i) *z is p-regular and D is a p-Sylow group of $C_G(z)$*; (ii) $D = Q^x \cap P = Q \cap P^y$, *and* (iii) $z = xy$.

To establish this, it is essential to extend the facts (1) through (6) above to cope with the situation of *two* p-Sylow groups. The point is that, because of the way the $G \times G$ action on Γ is defined, it is just as easy (*and obviously more effective*) to study the restriction of Γ and $\Gamma \cdot E$ to the subgroup $Q \times P \subset G \times G$. Fact (1) remains, of course, unchanged. For (2), we now have $\Delta(D) \subset Q \times P$, and, by Green's theory, the restriction $(\Gamma \cdot E)_{Q \times P}$ still has at least one indecomposable constituent with vertex $= \Delta(D)$. For (3), we must now let W be a full set of (Q, P)-double coset representatives in G (instead of (P, P)-coset representatives). *Notice that $Q \times P$ is a p-group*, and $[QwP]$ (with the obvious definition) is a transitive permutation module over $Q \times P$. By an earlier theorem of Green [2, Lemma 2.3a], $[QwP]$ *is still* $\mathfrak{O}(Q \times P)$*-indecomposable*. Hence, $\Gamma = \oplus_{w \in W}[QwP]$ remains a Krull-Schmidt decomposition in the category of $\mathfrak{O}(Q \times P)$-modules. For $x \in G$, let $J(x) = \{(a,b) \in Q \times P$: $axb^{-1} = x\}$, the subgroup of $Q \times P$ fixing x. (Green's $P(x,y)$ in [3, p.145] should be correspondingly changed into a more general $J(x,y)$, but we still have $J(x,y) = J(x) \cap J(y), \ldots,$ etc.) Letting $\{w_1, \ldots, w_c\}$ be a maximal subset of W such that $J(w_j)$ are mutually nonconjugate in $Q \times P$, we may assume, by what we said so far, that $\Delta(D) = J(w_1)$. Again, W may be 'normalized' so that each $J(w)$ $(w \in W)$ *equals* some (unique) $J(w_j)$.

The endomorphism algebra $\mathrm{End}_{Q \times P}\,\Gamma$, modulo its radical, can be calculated along essentially the same lines as in [3], with minor notational changes. The quotient breaks up into a product $\prod_{j=1}^{c} \mathbf{M}_{m_j}(\mathfrak{O}/\mathfrak{p})$, where the matrices in the jth factor are suffixed by the set $W_j = \{w \in W$: $J(w) = J(w_j) \subset Q \times P\}$.

By a procedure similar to that used in (5), one obtains again the representations Ψ_j. For the class sum K of any conjugacy class $\mathfrak{K}$, one can show that $\Psi_j(K)$ is evaluated as before, *after* replacing the old $F^j_{u,v}(\mathfrak{K}; P)$ by new sets:

$$F^j_{u,v}(\mathfrak{K}; Q, P) = \{x \in QuP\colon J(x) = J(w_j) \text{ and } xv^{-1} \in \mathfrak{K}\} \qquad (u, v \in W_j).$$

Finally, the new version of (6), proved by rather routine changes of Green's methods, now states that $\Psi_j(E)$ has rank equal to the number of indecomposable constituents of $(\Gamma \cdot E)_{Q \times P}$ with vertex conjugate in $Q \times P$ to $J(w_j)$.

Having said all the above, the proof of the Theorem now proceeds as follows (cf. [3, p.149]). Let $\mathfrak{K}_i$ $(1 \leqslant i \leqslant n)$ be the conjugacy classes of G, with sums K_i, and let $E = \sum_i^n a_i K_i$. Since $\Delta(D) = J(w_1)$, the new versions of (2) and (6) imply that $\Psi_1(E)$ has rank $\geqslant 1$. We have $0 \neq \Psi_1(E) = \sum_i \bar{a}_i \Psi_1(K_i)$ ('bar' denotes mod $\mathfrak{p}$), so there must exist a class $\mathfrak{K}_h$ for which $\bar{a}_h \neq 0$ and $\Psi_1(K_h) \neq 0$. The former implies (as is well known) that $\mathfrak{K}_h$ is p-regular, and the latter implies, thanks to the new version of (5), that the sets $F^1_{u,v}(\mathfrak{K}_h; Q, P)$ cannot be all empty for $u, v \in W_1$. Choose $u, v \in W_1$ for which there *exists* $x \in F^1_{u,v}(\mathfrak{K}_h; Q, P)$. We have then $J(x) = J(v) = J(w_1) = \Delta(D)$. But $J(x) = \{(a,b) \in Q \times P : a^x = b\}$. Thus, $J(x) = \Delta(D)$ implies that $D = Q^x \cap P$ and $x \in C_G(D)$. Similarly, we obtain $D = Q^v \cap P$ and $v \in C_G(D)$. Setting $y = v^{-1} \in C_G(D)$, we have $D = Q \cap P^y$. The element $z = xy = xv^{-1} \in \mathfrak{K}_h$ is, therefore, p-regular, and its class-defect group contains D. But then D must actually *equal* the class-defect group of $\mathfrak{K}_h$. Q.E.D.

Finally, from the Theorem just proved, we may record the following generalization of the 'tame-Sylow-intersection' property:

COROLLARY. *Let D be the defect group of a p-block of the finite group G, and let P be any p-Sylow group of G containing D. Then there exists a p-Sylow group S of G such that $D = S \cap P$, and $N_S(D)$ is a p-Sylow subgroup of $N_G(D)$.*

PROOF. Choose a p-Sylow group $Q \supset D$ such that $N_Q(D)$ is a p-Sylow group of $N_G(D)$. There exists, by the Theorem, $x \in C_G(D)$ such that $D = Q^x \cap P$. The new p-Sylow group $S = Q^x$ clearly has the desired properties. Q.E.D.

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DEPARTMENT OF MATHEMATICS, UNIVERSITY OF CALIFORNIA, BERKELEY, CALIFORNIA 94720