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A THEOREM ON NILPOTENT GROUPS WITH RESTRICTED EMBEDDINGS

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ABSTRACT. Suppose that G is a nonabelian group with a unique proper normal subgroup of some given order. It is proved that G is not contained as a normal subgroup within the Frattini subgroup of a finite p-group.

Burnside [1] has proved that nonabelian groups with cyclic center, or whose derived subgroup has index p^2 cannot occur as the derived subgroup of a p-group. Hobby [2] showed that the same classes of groups also cannot occur as the Frattini subgroup of a p-group. Nonabelian p-groups with cyclic center or whose derived subgroup has index p^2 are examples of groups which have a unique proper normal subgroup of some given order. In this note we prove that the theorems of Burnside and Hobby may be generalized to include this wider class of p-groups. In fact we prove the following more general theorem.[1]

THEOREM. *Suppose G is a nonabelian group with a unique proper normal subgroup of some given order. Then G is not contained as a normal subgroup within the Frattini subgroup of a finite p-group.*

PROOF. Denote by G_0 the unique proper normal subgroup of G. Suppose first that G_0 is cyclic and that G is a normal subgroup of a p-group P within the Frattini subgroup $\Phi(P)$. By [3, Satz III 7.5] there exists an elementary abelian subgroup R of G which is normal in P and has type (p,p). Since G_0 is cyclic it is contained in R and so has order p. As G_0 is the only normal subgroup of G having order p, it follows that $Z(G)$ is cyclic. However, since R is normal in P, the order of $P/C_P(R)$ divides the order of the group of automorphisms of R. Hence, $C_P(R)$ is either P itself or a maximal subgroup of P.

In either case $G \leqslant \Phi(P) \leqslant C_P(R)$ and, hence, $R \leqslant Z(G)$ contrary to the fact that $Z(G)$ is cyclic. Thus the theorem holds if G_0 is cyclic.

To complete the proof we use induction on the order of G_0. Suppose G_0 is noncyclic and G is a normal subgroup of a p-group P within the Frattini subgroup $\Phi(P)$. Let L be a normal subgroup of P of order p contained in G_0; then G_0/L is a unique proper normal subgroup of G/L of the given order $|G_0/L| < |G_0|$. Also $G/L \leqslant \Phi(P)/L = \Phi(P/L)$ and by induction G/L is abelian and, hence, cyclic. Since $L \leqslant Z(G)$, G is abelian, contrary to the hypothesis of the theorem. This completes the proof of the theorem.

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