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RESIDUAL SOLVABILITY OF AN EQUATION IN NILPOTENT GROUPS

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ABSTRACT. Let G be a finitely generated nilpotent group. Let S_1 and S_2 be subgroups of G. Let S_1S_2 be the set of all products g_1g_2, where g_i is an element of S_i. Let g be an element of G. It is shown that either g is an element of S_1S_2 or there is a normal subgroup N of finite index in G such that gN does not meet S_1S_2. This result implies:

(a) There is an algorithm to determine whether or not g is an element of S_1S_2.

(b) Given elements a, b, and c of G, there is an algorithm to determine whether there exist integers n and m such that $a = b^mc^n$.

(c) Finitely generated nilpotent groups are subgroup separable (a result of K. Toh).

(d) Given elements a and b of G and a subgroup S of G, there is an algorithm to determine whether or not a is an element of SbS.

Recall that every finitely generated nilpotent group G is polycyclic, i.e. there exists a series N_i such that $G = N_0 \supset N_1 \supset \cdots \supset N_k = 1$, where N_{i+1} is a normal subgroup of N_i and the factor groups N_i/N_{i+1} are cyclic. If N_i and M_i are any two such polycyclic series for G, it follows from inspection of a common refinement of the series that the number of infinite cyclic factor groups is independent of the series selected. The number of infinite cyclic factor groups for any polycyclic series of G is called the torsion free rank of G. Clearly, if C is an infinite cyclic normal subgroup of G, the torsion free rank of G/C is one less than that of G. Recall also that if G is an infinite finitely generated nilpotent group, there is an element of infinite order in the center of G. These remarks allow us to use induction on the torsion free rank of g, as in the monograph by G. Baumslag [1]. This reference is the general reference for theorems on nilpotent groups.

THEOREM. *Let G be a finitely generated nilpotent group. Let S_1 and S_2 be subgroups of G. If g is an element of G, then either g is an element of S_1S_2 or there is a normal subgroup N of finite index in G such that gN does not meet S_1S_2.*

PROOF. Let Z be the center of G. Note that if z is an element of S_1S_2 and an element of Z, then the cyclic subgroup generated by z is contained in S_1S_2. Let $z = s_1s_2$, where s_i is contained in S_i. We have $s^{-1}z = s_2$, so that $(s_1^{-1}z)^k = s_2^k$ for all integers k. Since z is central, $s_1^{-k}z^k = s_2^k$, and so $z^k = s_1^ks_2^k$.

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The proof of the theorem is by induction on the torsion free rank of G. If G has torsion free rank zero, G is finite and the result follows with N the identity subgroup. Assume G has positive torsion free rank and that the result is valid for groups of lesser torsion free rank. Let z be an element of infinite order in Z. If, for a given nonzero integer k, g is not contained in $S_1S_2\langle z^k\rangle$, the result follows, for $\langle z^k\rangle$ is normal in G and $G/\langle z^k\rangle$ has lesser torsion free rank than G. (Note that $\langle z^k\rangle$ is the subgroup generated by z^k.) We can thus assume that g is an element of $S_1S_2\langle z^k\rangle$ for all nonzero k. It will follow that this implies that g is an element of S_1S_2.

Since $g \in S_1S_2\langle z\rangle$, there are elements s_1 and s_2 of S_1 and S_2, respectively, and an integer i such that $g = s_1s_2z^i$. Since $g \in S_1S_2\langle z^{1+i^2}\rangle$, there are elements c_1 and c_2 of S_1 and S_2 and an integer u such that $g = c_1c_2z^{u(1+i^2)}$. If either i or u is zero, we are finished. If i and u are not zero, $v = u(1+i^2) - i$ is not zero, since i and $1 + i^2$ are relatively prime. Equating the two expressions for g and recalling that z is central, $c_1^{-1}s_1s_2c_2^{-1} = z^{u(1+i^2)-i}$.Thus z^v is a central element of S_1S_2. By the lemma at the start of the proof, the entire cyclic subgroup generated by z^v is in S_1S_2. Since v is not zero, it follows by assumption that

$$g \in S_1S_2\langle z^v\rangle = S_1\langle z^v\rangle S_2 \subset S_1S_1S_2S_2 = S_1S_2.$$

REMARKS. The result (a) follows from standard methodsof logic and the restatement of the result is as follows: Let g be an element of G. Either G is in the complex S_1S_2 or there is a homomorphism θ of G onto a finite group such that $g\theta \not\in S_1\theta S_2\theta$.

The result (b) is the case of S_1 and S_2 cyclic. This case has been considered in the study of other groups [2].

Subgroup separability is the case of one of the subgroups being the identity group. This result was obtained by K. H. Toh [3].

The problem solved in (d) is called the double coset problem. We replace the formula $a \in SbS$ by the equivalent formula $ab^{-1} \in SbSb^{-1}$ and set $S_1 = S$, $S_2 = bSb^{-1}$ in the theorem.

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