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A PROPERTY OF GROUPS OF NONEXPONENTIAL GROWTH

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ABSTRACT. We prove that in a finitely generated nonexponential group a normal subgroup with a solvable quotient is finitely generated. This extends a theorem of Milnor which has the same conclusion if the group is also assumed to be solvable. The proof uses a lemma of Milnor, but in a different, simpler, way.

We prove a theorem which is an extension of a result of J. Milnor.

THEOREM. *If G is a finitely generated group which does not grow exponentially and H is a normal subgroup such that G/H is solvable, then H is finitely generated.*

The basic tool is recorded in §2 as 'Milnor's lemma', and it shows that finitely generated nonexponential groups have a certain property. Indeed the Theorem above could be stated for groups having this property.

Milnor stated the lemma in [3] and used it to prove that a finitely generated, nonexponential solvable group is polycyclic. This is equivalent, of course, to $G' = [G, G]$ being finitely generated, for then polycyclicity follows by induction on the derived length of G. The Theorem above extends Milnor's result in that it shows that G' is finitely generated without the assumption that G is solvable.

1. **Definitions and background.** If G is a finitely generated group, and $x_1, x_2, \ldots, x_n$ generate it, the growth function of G, with respect to this set of generators is defined to be

$$g(s) = \text{number of elements in } G \text{ expressible as words of length } \leqslant s \text{ in the generators and their inverses}.$$

The function g is said to have exponential growth if for some real numbers $c > 0, a > 1, g(s) \geqslant c \cdot a^s$ for every integer $s \geqslant 1$. Otherwise g is 'nonexponential'. If g is bounded by some polynomial of degree m we say that it grows polynomially. Growth functions $f(s)$, $g(s)$ are said to be equivalent if there exist integers $A, B > 0$ such that, for every $s \geqslant 1, f(s) \leqslant g(As), g(s) \leqslant f(Bs)$. It is easy to see that all growth functions of G are equivalent, that equivalent functions have the same type of growth and, in the case of polynomial growth, that equivalent functions are bounded by polynomials of the same degree. Thus growth type and degree are invariants of the group.

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Growth functions were first introduced by Milnor [2] in a geometric context. Wolf [6] showed that finitely generated polycyclic groups are either exponential or have a nilpotent subgroup of finite index, and, therefore, polynomial growth. A purely algebraic proof of this, using only Milnor's lemma, is in [1]. Milnor's [3] was intended to extend Wolf's result from 'polycyclic' to 'solvable'.

It is not known if there exist groups which are not exponential and not polynomially growing. Wolf conjectured that polynomiality implies almost solvability and Bass extended the conjecture to the nonexponential case. (A group is 'almost P' if it has a P-subgroup of finite index.) A theorem of Tits [5] affirms the conjecture for subgroups of linear groups.

2. In this section we prove the Theorem and given some comments.

LEMMA 1 (MILNOR [3]). *If G is a finitely generated nonexponential group, and if $x, y \in G$, then the group generated by the set of conjugates y, xyx^{-1}, x^2yx^{-2}, $\ldots$ is finitely generated.*

Note that Milnor stated the lemma under the restriction that all the conjugates commute, but his proof does not use this restriction. For the sake of completeness (and reassurance) we repeat the proof: The number of words $xy^{i_1}xy^{i_2}x\cdots xy^{i_m}$, where each $i_j = 0$ or 1, is 2^m. G being nonexponential they cannot all be distinct for all m. Let $y_i = x^iyx^{-i}$. For the minimal m for which some equality occur:

$$xy^{i_1}xy^{i_2}\cdots xy^{i_m} = xy^{j_1}xy^{j_2}\cdots xy^{j_m}, \qquad i_m \neq j_m.$$

So

$$y_1^{i_1}y_2^{i_2}\cdots y_m^{i_m} = y_1^{j_1}y_2^{j_2}\cdots y_m^{j_m},$$

and $j_m - i_m = \pm 1$ implies that $y_m = W(y_1, \ldots, y_{m-1})$. But then

$$y_{m+1} = W(y_2, \ldots, y_m) = W'(y_1, \ldots, y_{m-1}),$$

etc. and the proof is now clear.

The next lemma is the key to the whole proof and it uses the following well-known fact: let X be a generating set of a group G, H a subgroup and T a set of representatives of the right cosets of H, with $1 \in T$. Then H is generated by the set $TXT^{-1} \cap H$. See [4, p. 152] for a proof.

LEMMA 2. *Let G be a finitely generated group in which the conclusion of Lemma 1 holds. Suppose $H \rhd G$ and G/H is infinitely cyclic. Then H is finitely generated.*

PROOF. Let t be a generator of G/H. We can find a generating set $\{x_1, x_2, \ldots, x_k\} = X$ such that x_1 maps into t and $x_2, \ldots, x_k \in H$. Indeed, if x_i maps onto t^{m_i}, we can modify it by a power of x_1, $x_1^{-m_i}x_i$, and x_1, x_i generate the same group as x_1, $x_1^{-m_i}x_i$. We take for a set of representatives the cyclic subgroup generated by x_1. Then

$$x_1^i x_l x_1^j = x_1^i x_l x_1^{-i} x_1^{i+j} \in H \Leftrightarrow i + j = 0,$$

that is, H is generated by all the conjugates of $x_2, x_3, \ldots, x_k$ by x_1. Milnor's lemma now implies that H is finitely generated.

LEMMA 3. *G as in Lemma 2, $H \rhd G$ and G/H abelian. Then H is finitely generated.*

PROOF. Write G/H as a direct sum of a torsion free part and the torsion. The inverse image of the torsion free part has finite index in G, so is finitely generated, and, modulo H, is torsion free. So we can assume that $G/H = F$ is free abelian. If rank $F = 1$, this is Lemma 2. Assume the lemma for rank $F < n$ $(n \geqslant 1)$. If rank $F = n$, write $F = A + B$, where A has rank $n - 1$ and B rank 1. The inverse image of A is finitely generated by Lemma 2, and the finite generation of H now follows from the inductive assumption.

COROLLARY 1. *G as above. Then G' is finitely generated.*

COROLLARY 2. *If G is also assumed solvable, then G is polycyclic.*

Recall that a group is polycyclic if it is solvable and all its derived subgroups are finitely generated.

We can now complete the proof of the Theorem: use induction on the derived length of G/H. If the derived length of G/H is 1, G/H is abelian and this is Lemma 3. Assuming the result for derived length $n - 1$, let L be the inverse image of $[G/H, G/H]$ in G. L is finitely generated by Lemma 3 and L/H has derived length $n - 1$. The Theorem follows.

As for the problem mentioned at the end of §1, we can use a statement of Bass [1] to give a positive answer in a very special case. In the derived series, if $(G^{(n)} : G^{(n+1)}) < \infty$, we say that it is a finite step. Bass shows that if G has polynomial growth of degree d and if $H_0 < H_1 < \cdots < H_n$ are finitely generated and $(H_i : H_{i-1}) = \infty$ for $1 \leqslant i \leqslant n$, then $n \leqslant d$. Since we have proved that all the derived groups are finitely generated, we deduce that the number of nonfinite steps in the derived series is finite. In particular, if the growth is polynomial and all steps are known to be infinite, it follows that G is solvable.

I do not know of any example of a nonexponential group with a properly descending derived series such that $1 < (G^{(n)} : G^{(n+1)}) < \infty$ at every step. A simple example of a group with polynomial growth such that $(G : G') < \infty$ is the free product $\mathbf{Z}_2 * \mathbf{Z}_2$ which grows linearly but G_{ab} is of order 4.

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