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CERTAIN MULTIPLE VALUED HARMONIC FUNCTION

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ABSTRACT. The difference equation $v(x^+) - v(x) = u(x)$ is solved for any harmonic u in the covering space of an unknotted curve.

The purpose of this note is to answer a problem posed by H. Lewy [2]. The situation is as follows: Given is a curve Γ diffeomorphic, in $\mathbf{R}^3$, to a circle. That is, we have a diffeomorphism $\varphi: \mathbf{R}^3 \to \mathbf{R}^3$ that maps Γ onto a circle C.

If we now consider the universal covering space S, of $\mathbf{R}^3 \sim \Gamma$, it is asked if given a function u, harmonic on S, we can find a harmonic function v, such that $v(x^+) - v(x) = u(x)$ (where x^+ denotes the point obtained from x in S following a path δ whose projection $\pi(\delta)$ in $\mathbf{R}^3 \sim \Gamma$ is a closed curve that loops Γ once). We are going to prove that such a function v exists.

PROOF. The proof is a modification of the proof in [2], to make an integral convergent.

More precisely, let us consider, as in [2], the disk D in $\mathbf{R}^3$ that has C as boundary and let $\sigma = \varphi^{-1}(D)$. We will consider neighborhoods U_n of Γ,

$$U_n = \{x: d(x, \Gamma) \leqq 2^{-n}\}. \tag{1}$$

By means of the diffeomorphism, for n_0 large enough we can construct a continuous mapping $\tilde{x}(x): U_{n_0} \to \Gamma$ such that

$$|\tilde{x}(x) - x| \leqq K d(x, \Gamma) \qquad (K \text{ a constant}). \tag{2}$$

B_1 will denote a ball verifying $(U_0 \cup \sigma) \subset B_1$.

Let us consider now the two consecutive leaves S_1, S_2 of S that are between the copies σ_0 and σ_2 of σ on S and have the copy σ_1 as common boundary.

Then, in the compact set

$$W_n = [(S_1 \cup S_2) \cap \pi^{-1}(B_1)] \sim \pi^{-1} U_n$$

there exists a λ_n such that

$$|u| \leqq \lambda_n. \tag{3}$$

(As before π denotes the canonical projection $\pi: S \to \mathbf{R}^3 \sim \Gamma$.) Any first derivative of u is therefore bounded in W_n by some λ_n'. Let

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$$\bar{\lambda}_n = \max(\lambda_n, \lambda_n'). \tag{4}$$

We are going to consider the function

$$\begin{aligned} v(x) = \frac{1}{4\pi} \int_{\sigma \sim U_{n_0}} \left(u(y) \frac{\partial}{\partial \nu_y} V - V \frac{\partial}{\partial \nu_y} u(y) \right) d\sigma(y) \\ + \sum_{k=n_0}^{\infty} \int_{\sigma \cap (U_k \sim U_{k+1})} \left(u(y) \frac{\partial}{\partial \nu_y} (V - V_k)(x,y) \right. \\ \left. -(V - V_k)(x,y) \frac{\partial}{\partial \nu_y} u(y) \right) d\sigma(y) \end{aligned} \tag{5}$$

where $V(x,y) = 1/|x-y|$ and V_k is a suitable Taylor's polynomial of V around the point $\tilde{y}$ (as defined in (2)). The values of u are taken over the copy σ_1 of σ, bounding S_1 from S_2. Let us then recall the following developments:

$$\frac{1}{|x-y|} = \frac{1}{|x-y_0|} \left(\sum_0^{\infty} P_n(\cos\theta) \left(\frac{|y-y_0|}{|x-y_0|} \right)^n \right) \tag{6}$$

convergent for $|y - y_0| < |x - y_0|$, $\sup_{|u| \leqq 1} |P_n(u)| = 1$ with θ the angle between $x - y_0$ and $y - y_0$.

$$\begin{aligned} \nabla_y \frac{1}{|x-y|} = \frac{1}{|x-y_0|^3} \sum_1^{\infty} \left(P'_{n-1}(u)(y-y_0) - P'_n(u) \frac{|y-y_0|}{|x-y_0|} (x-y_0) \right) \\ \cdot \left(\frac{|y-y_0|}{|x-y_0|} \right)^{n-2} \end{aligned} \tag{7}$$

with $u = \cos\theta$ again convergent for $|y - y_0| < |x - y_0|$, $\sup_{|u| \leqq 1} |P'_n(u)| \leqq n(n+1)/2$.

Each term of (6) and (7) is a harmonic polynomial (see [1, pp. 124 and 142]).

Turning back to (5) we choose V_k to be the development (6) with $y_0 = \tilde{y}(y)$ up to an order $l_{(k)}$ verifying

$$\bar{\lambda}_k 2^{-l_{(k)}/2} < 2^{-k}. \tag{8}$$

Then if $x \notin U_m \cup \sigma_1$ and $B(x)$ is a ball centered at x with $\overline{B(x)} \cap (U_m \cup \sigma_1) = \varnothing$, for $y \in U_k$, with $k \geqq \max[K(m+1), n_0]$, we have, for any $x' \in B(x)$,

$$(V - V_k)(x', y) \leqslant \frac{C 2^{-l_{(k)}/2}}{|x' - \tilde{y}|} \leqslant C 2^m 2^{-l_{(k)}} \tag{9}$$

and

$$|\nabla(V - V_k)|(x', y) \leqq 2^{3m} C l_{(k)}^2 2^{-l_{(k)}} \leqq C' 2^{-l_{(k)}/2} \tag{10}$$

where C and C' denote constants. Hence, the first $\max[K(m+1), n_0]$ terms of (5) are bounded harmonic functions on B, and the remaining terms give us, on B, an absolutely and uniformly convergent series of harmonic functions. Now, if B is a small ball intersecting σ (with $\bar{B} \cap \Gamma = \varnothing$), and we remove from (5) the integral

$$\frac{1}{4\pi}\int_{B\cap\sigma}\left(u(y)\frac{\partial}{\partial\nu_y}V(x,y) - V(x,y)\frac{\partial}{\partial\nu_y}u(y)\right)d\sigma \tag{11}$$

(which involves only a finite number of terms of (5)), the remaining terms give us again an absolutely and uniformly convergent series of harmonic functions in any closed subball of B. But as in [2], (11) gives us the desired jump in $u(x)$ of the function v across σ, and that completes the proof.

References

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