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SIMULTANEOUS SPLINE APPROXIMATION AND INTERPOLATION PRESERVING NORMS

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ABSTRACT. In this paper, it is proved that splines of order k $(k \geqslant 2)$ have property SAIN. The proof of this result is based on the important properties of B-splines.

1. **Introduction.** In a recent manuscript [5], Lambert proved that the twice continuously differentiable cubic splines possess property SAIN (simultaneous approximation and interpolation which is norm preserving) on $C[a, b]$ where the interpolatory constraints are point evaluations. In this paper we establish the more general result for splines of any order greater than 1 while at the same time supplying a simple proof. More precisely, we will show

THEOREM 1. *Splines of order k (degree $k - 1$) and continuity class $C^{k-2}[a, b]$ possess property SAIN on $C[a, b]$ with respect to a finite number of point evaluations.*

Deutsch and Morris [2] introduced the property SAIN:

DEFINITION. *Let X be a normed linear space, M a dense subset of X, and Γ a finite dimensional subspace of X^*. The triple (X,M,Γ) has property SAIN if, for every $x \in X$ and $\varepsilon > 0$, there exists $y \in M$ such that*

$$\|x - y\| < \varepsilon, \quad \|x\| = \|y\|, \quad \textit{and} \quad \gamma(x) = \gamma(y)$$

for every $\gamma \in \Gamma$.

This definition was motivated by the work of Wolibner [6] and Yamabe. (See [2] for references.) Also Lambert [4] has studied property SAIN for L_1 and $C(T)$, and Holmes and Lambert [3] studied the property SAIN from a geometrical point of view.

2. **Proof of Theorem 1.** Let S^k denote the set of splines of order k and continuity class C^{k-2} with a finite number of knots in $[a, b]$. Further set $\Gamma = \operatorname{span}[\delta_{\tau_1}, \dots, \delta_{\tau_l}]$ where δ_x represents the usual point evaluation functional at x which is an element of $C[a, b]^*$. We now show that $(C[a, b], S^k, \Gamma)$ has property SAIN.

Our proof relies heavily on the fundamental properties of B-splines. Following de Boor [1], we denote by $N_{i,k}$ the normalized B-spline of order k supported on $[t_i, t_{i+k}]$ where $\{t_i\}_{i=0}^N$ is a partition of $[a, b]$, and where $N_{i,k}(t) \equiv (t_{i+k} - t_i)[t_i, \dots, t_{i+k}]_S(S - t)_+^{k-1}$ with $[t_i, \dots, t_{i+k}]_S$ denoting the

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kth divided difference operator in the variable S. Recall the following important properties of the $N_{i,k}$:

(i) $N_{i,k}(t) \geqslant 0$ for all t,
(ii) supp $N_{i,k} = [t_i, t_{i+k}]$, and
(iii) $\sum_{i=j+1-k}^{j} N_{i,k}(t) \equiv 1$, $t \in [t_j, t_{j+1}]$.

(For a detailed account of B-splines, see [1] .) Let π denote a partition of $[a, b]$, i.e., $\pi = \{t_i\}_{i=0}^{N}$ with $a = t_0 < \cdots < t_N = b$. The mesh size of π will be denoted by $|\pi| \equiv \max_{0 \leqslant i \leqslant N-1}(t_{i+1} - t_i)$. In order for the $N_{i,k}$'s to be a basis of the C^{k-2} (order k) splines, with knots only in π, it is necessary to create $k - 1$ knots on the left of t_0 and $k - 1$ knots on the right of t_N. Given a partition π of $[a, b]$ we define $\tilde{\pi}$ as the partition $t_{-k+1} < \cdots < t_{N+k-1}$ where the extra points are chosen so that $|\tilde{\pi}| = |\pi|$.

Proceeding with the proof, let $f \in C[a, b]$ be given along with interpolation points $\{\tau_i\}_{i=1}^{l}$ and $\varepsilon > 0$. Without loss of generality assume that f attains its norm at some τ_i. It remains to display a spline of the correct type which interpolates f at the τ_i, is within ε of f in the supremum norm, and which is norm preserving. Pick $\delta > 0$ so that the modulus of continuity ω of f satisfies $\omega(f, \delta) \leqslant \varepsilon/2k$ and let π (and hence $\tilde{\pi}$) be chosen so that $|\pi| = |\tilde{\pi}| \leqslant \delta$ and $(k + 1)|\tilde{\pi}| \leqslant \min_{i \neq j}|\tau_i - \tau_j|$. Define

$$g(t) = \sum_{i=-k+1}^{N-1} \alpha_i N_{i,k}(t)$$

where $\alpha_i = f(\tau_m)$ for $i = j - k + 1, \ldots, j$ whenever $t_j \leqslant \tau_m < t_{j+1}$, $m = 1, \ldots, l$; and $\alpha_i = f(t_i)$ otherwise, provided that we define $f(t_i) \equiv f(a)$ for those $t_i < a$. It is clear from (i), (ii), and (iii) that $\|g\| = \|f\|$, g interpolates f at the τ_i and

$$\|g - f\| = \left\| \sum_{i=-k+1}^{N-1} \alpha_i N_{i,k}(t) - f(t) \right\| = \left\| \sum_{i=-k+1}^{N-1} (\alpha_i - f(t)) N_{i,k}(t) \right\|$$
$$\leqslant \max_{-k+1 \leqslant i \leqslant N-1} \max_{t \in B_i} |\alpha_i - f(t)| \leqslant \omega(f, 2k\delta) \leqslant 2k\omega(f, \delta) \leqslant \varepsilon,$$

where $B_i \equiv [t_i, t_{i+k}] \cap [a, b]$. This completes the proof of the theorem.

REMARK. In the course of the proof of Theorem 1, we add, if necessary, an additional interpolation constraint at a point where f attains its norm. This is to insure that g satisfies the norm preservation property. However, even without adding this additional interpolation constraint, we still have $\|g\| \leqslant \|f\|$, and the theorem then follows by applying Lemmas 2.1 and 2.2 of [2].

There are many ways to extend Theorem 1. For instance, one can consider the natural splines

$$S_0^{2k} = \{s \in S^{2k} : s^{(j)}(a) = s^{(j)}(b) = 0,\ k \leqslant j \leqslant 2k - 2\}$$

and obtain the following:

COROLLARY. *The natural splines S_0^{2k} have the property SAIN.*

To prove this result, we just alter the construction above to insure that $\alpha_{-k+1} = \cdots = \alpha_0$ and $\alpha_{N-1} = \cdots = \alpha_{N-k}$, so that $g^{(j)}(a) = g^{(j)}(b) = 0$ for $j = k, \ldots, 2k - 2$.

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