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A CHARACTERISATION OF DISCRETENESS FOR LOCALLY COMPACT GROUPS IN TERMS OF THE BANACH ALGEBRAS $A_p(G)$

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ABSTRACT. The Banach algebra A is said to have the bounded power property if for any $x \in A$, with $\|x\|_{sp} = \lim_{n\to\infty}\|x^n\|^{1/n} \leq 1$, one has $\sup_n\|x^n\| < \infty$. It has been shown by B. M. Schreiber [9, Theorem (8.6)] that, if G is a locally compact abelian group, then the Fourier algebra $A(G) = L^1(\Gamma)\hat{}$ has the bounded power property, if and only if G is discrete. We improve this result in the THEOREM. *Let G be an arbitrary locally compact group and $1 < p < \infty$. Then $A_p(G)$ has the bounded power property if and only if G is discrete.* Our proof, even for abelian G and $p = 2$ (then $A_2(G) = A(G)$ is the usual Fourier algebra of G), is much simpler and entirely different from that of [9].

The bounded power property for $L^1(G)$ with usual convolution (as in [7]) was investigated thoroughly by B. Schreiber [9]. Among other results, Schreiber obtains [9, Theorem (8.6)] the following result: (S) If G is locally compact abelian then $L^1(G)$ has the bounded power property if and only if G is compact.

In fact Schreiber proves more than that. He shows, among many other results, that if G belongs to a class $\mathcal{G}$ of locally compact groups, which contains all abelian and all compact groups and all groups, all of whose unitary irreducible representations are finite dimensional, then $L^1(G)$ has the bounded power property if and only if G is compact and abelian. He conjectures this result to be true for any locally compact G.

Schreiber's result (S) can be restated as follows: If G is abelian, $\Gamma = \hat{G}$, then $A(\Gamma)$ has the bounded power property iff Γ is discrete.

C. Herz has introduced and studied (see [5], [6]) for any locally compact G and any $1 < p < \infty$, the function algebras $A_p(G)$. In case $p = 2$, $A_p(G)$ coincides with the Fourier algebra $A(G)$ studied extensively by P. Eymard [2] and in case G is also abelian it coincides with the usual Fourier $A(G) = L^1(\Gamma)\hat{}$. We prove in this paper the following:

THEOREM A. *Let G be any locally compact group and $1 < p < \infty$. Then $A_p(G)$ has the bounded power property iff G is discrete.*

We should point out that if G is discrete, then $A_p(G)$ is a semisimple Banach algebra with discrete maximal ideal space, hence this part of Theorem A follows immediately from Corollary (2.3) of Schreiber [9, p. 408]. We give however, for the sake of completeness, an immediate proof of this part too.

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We have the following additional remarks:

(1) Schreiber's proof of (S) uses: P. J. Cohen's idempotent theorem, P. J. Cohen's theorem on homomorphisms of group algebras which are implemented by piecewise affine maps ([1], or [8, Theorem 4.1.3, p. 78]) (both, in the proof of Lemma (6.1) of [9, pp. 415–416]), and J. E. Gilbert's [4] and B. M. Schreiber's [10] result on the representation of closed sets in the coset ring (in the proof of Theorem (6.2) [9, p. 416] which in turn is used in Lemma (8.5), hence in [9, Theorem (8.6), p. 429]). Our proof of Theorem A is simple and when restricted to abelian G and $p = 2$ (i.e. to $A(G)$) uses only basic facts taught in any first course in harmonic analysis.

(2) Our Theorem A does not throw any light on Schreiber's conjecture mentioned above, but only reduces to the statement (S) in case $p = 2$ and G is abelian.

We express our thanks to M. Cowling for pointing out to us the paper [6] by C. Herz.

Notations. λ will denote a left Haar measure on G and we follow the basic notations of [7]. In particular, the $L^p(G)$ norm will be denoted by $\|f\|_p = (\int |f|^p \, d\lambda)^{1/p}$. If $1 < p < \infty$ let p' be defined by $1/p + 1/p' = 1$. For the algebra $A_p(G)$ we follow C. Herz [5]. In particular, $f \in A_p(G)$ iff $f = \sum_1^\infty v_n * \tilde{u}_n$ (absolutely and uniformly convergent sum) with $\sum \|u_n\|_p \|v_n\|_{p'} < \infty$. $\|f\|_{A_p}$ will denote the infimum of these last sums over all such representations of f. Herz proves in [5], among other results, that $(A_p(G), \| \ \|_{A_p})$ is a Banach algebra with respect to pointwise multiplication whose maximal ideal space is G (and which is a regular, tauberian algebra of functions on G [5, pp. 100–102]). Clearly $\|f\|_\infty \leqslant \|f\|_{A_p}$ for $f \in A_p(G)$.

If $S \subset G$, 1_S is defined by $1_S(x) = 1$ if $x \in S$ and equals 0 at all other $x \in G$.

PROPOSITION 1. *Let G be a second countable locally compact group and $1 < p < \infty$. If $A_p(G)$ has the bounded power property then G is discrete.*

PROOF. Let V be a symmetric relatively compact neighborhood of e the unit of G. Consider the function

$$\phi_V(x) = 1_V * 1_V^{\sim}(x) = \int 1_V(x^{-1}y) 1_V(y) \, dy = \lambda(xV \cap V).$$

Then $0 \leqslant \phi_V(x) \leqslant \lambda(V) = \phi_V(e)$. Since

$$\phi_V(e) \leqslant \|\phi_V\|_{A_p} \leqslant \|1_V\|_p \|1_V\|_{p'} = \lambda(V)$$

one has that $\phi_V(e) = \|\phi_V\|_{A_p}$. Let $\psi_V = \lambda(V)^{-1}\phi_V$. Then

$$\|\psi_V\|_{A_p} = 1 = \psi_V(e) \quad \text{and} \quad 0 \leqslant \psi_V(x) \leqslant 1.$$

Moreover, $\psi_V = 0$ off V^2 since so does ϕ_V. It follows that $V_1 = \{x;\ \psi_V(x) > 0\} \subset V^2$ and, in particular, that for any neighborhood U of e there exists a relatively compact open neighborhood V_1 of e such that $V_1 \subset U$ and for some $\psi \in A_p(G)$ with $0 \leqslant \psi \leqslant 1$, $\|\psi\| = 1 = \psi(e)$ one has $V_1 = \{x;\ \psi(x) > 0\}$.

Let $\mathcal{V}$ be a neighborhood base at e in G consisting of sets V_1 with this property.

Let K be any closed set in G and $W = G \sim K$. Then, for every $a \in W$ there is some $V \in \mathcal{V}$ such that $aV \subset W$. Thus $W = \bigcup_1^\infty a_n V_n$ where $V_n \in \mathcal{V}$ and $a_n \in W$ [11, p. 49]. Let $\psi_n = \psi_{V_n}$ be corresponding functions in $A_p(G)$ for V_n and let $\psi = \sum_1^\infty 2^{-n} l_{a_n} \psi_n$ where $(l_a f)(x) = f(a^{-1}x)$. $\psi \in A_p(G)$, since $\|l_{a_n}\psi_n\|_{A_p} = \|\psi_n\|_{A_p} = 1$. Clearly $0 \leqslant \psi \leqslant 1$, $\psi(x) > 0$ for all $x \in W$ and $\psi(x) = 0$ for all $x \in K$. We have shown that for any closed $K \subset G$ there exists $\psi \in A_p(G)$ such that $0 \leqslant \psi \leqslant 1$ and $K = \psi^{-1}(0)$.

From now on, let $K \subset G$ be compact, nowhere dense, and such that $\lambda(K) > 0$. If G is nondiscrete, such K exists. Let $\psi \in A_p(G)$ be such that $K = \psi^{-1}(0)$ and $0 \leqslant \Psi \leqslant 1$. Let $u \in A_p(G)$ be such that $0 \leqslant u \leqslant 1$ and $u(x) = 1$ for all $x \in K$. (Take as usual $u = \lambda(V)^{-1}[1_{KV} * 1^{\sim}{}_V]$ where V is any relatively compact symmetric neighborhood of e.) Then $\phi = u(1 - \psi) = u - u\psi \in A_p(G)$. Moreover $\{x;\ \phi(x) = 1\} = \{x;\ \psi(x) = 0\} = K$, i.e. $\phi^{-1}(1) = K$. Our assumption implies now that $\sup\|\phi^n\| < \infty$. Thus, $\{\phi^n;\ n \geqslant 1\}$ is a w^* compact subset of the Banach algebra $B_p(G)$, which is the Banach space dual of the normed space $L^1(G)$ with the norm QF_p (which is stronger than PF_p, the norm on $L^1(G)$ acting as convolution operators on $L^P(G)$. In case G is amenable, $B_p(G)$ is the dual of $L^1(G)$ with the PF_p norm. See C. Herz [6, Proposition 2 and the remarks thereafter]). Thus there exists some $v \in B_p(G)$ (which consists only of bounded continuous functions (Herz [6, Proposition 3])) such that $\int \phi^{n_\alpha}(x) f(x)\, dx \to \int v(x) f(x)\, dx$ for all $f \in L^1(G)$. But

$$\lim_{n\to\infty} \int \phi^n(x) f(x)\, dx = \int 1_K(x) f(x)\, dx \quad \text{for all } f \in L^1(G).$$

Hence $1_K(x) = v(x)$ a.e. Thus $v^2(x) = v(x)$ a.e. and since $v(x)$ is continuous, $v^2(x) = v(x)$ for all x. Thus $v(x) = 1_{K_1}(x)$ for some open and closed $K_1 \subset G$ such that $1_K = 1_{K_1}$ a.e. But $K_1 \sim K$ is open and $\lambda(K_1 \sim K) = 0$. Thus $K_1 \subset K$. Since $\lambda(K_1) = \lambda(K) > 0$, K_1 is nonvoid, which contradicts the fact that K is nowhere dense.

REMARK. If G is abelian and $p = 2$, then $B_2(G) = M(\Gamma)^\wedge$, and we used only the fact that $B_2(G)$ is a dual space (to $C_0(\Gamma)$) and its unit ball is w^* compact.

THEOREM. *Let G be any locally compact group, $1 < p < \infty$. If $A_p(G)$ has the bounded power property then G is discrete.*

PROOF. Let $G_0 = \bigcup_1^\infty U^n$ where U is a symmetric relatively compact neighborhood of e in G. Let $\phi_0 \in A_p(G_0)$ be such that $|\phi_0(x)| \leqslant 1$ for all x. Let $\phi(x) = \phi_0(x)$ for $x \in G_0$ and $\phi(x) = 0$ for other $x \in G$. Then by Herz [5, p. 106] $\phi \in A_p(G)$ and $\|\phi_0^n\|_{A_p(G_0)} \leqslant \|\phi^n\|_{A_p(G)}$. We can hence (and shall) assume that G is compactly generated (since if G_0 is discrete so is G). Assume that G is not discrete and let V_n be a sequence of relatively compact neighborhoods of e such that $\lambda(V_n) \to 0$ and let $K \subset \bigcap_1^\infty V_n$ be a compact normal subgroup such that $G_1 = G/K$ is separable metric. Let $\phi_1 \in A_p(G_1)$ be such that $|\phi_1(x^1)| \leqslant 1$ for all $x^1 \in G_1$ and let $\phi(x) = \phi_1(x^1)$ where $x^1 \in G_1$ represents the coset xK. Then, by Herz [5, p. 106] $\|\phi_1^n\|_{A_p(G_1)} \leqslant \|\phi^n\|_{A_p(G)}$. Clearly, $|\phi(x)| \leqslant 1$ for all x, hence, $\sup\|\phi^n\|_{A_p(G)} < \infty$. This shows that $A_p(G_1)$ has the bounded power property. By Proposition 1, G_1 is discrete, which implies that K is open. But $\lambda(K) = 0$ which cannot be. Hence,

there does not exist in G a sequence of neighborhoods of e, V_n, such that $\lambda(V_n) \to 0$, i.e. G is discrete.

For the sake of completeness we give a short easy proof of the converse of Theorem 1. This converse is due to B. M. Schreiber [9, Corollary 2.3, p. 408].

PROPOSITION 2. *Let G be discrete. Then $A_p(G)$ has the bounded power property.*

PROOF. Let $f_1 \in A_p(G)$ have finite support and $|f_1(x)| \leqslant 1$ for all x. Then $f_1 = \sum_1^k f(a_i)\delta_{a_i}$ where δ_{a_i} is the unit mass at a_i (hence $\|\delta_{a_i}\|_{A_p} = \|\delta_e\|_{A_p} = 1$). Then

$$\|f_1^n\|_{A_p} = \left\| \sum_{i=1}^{k} f_1(a_i)^n \delta_{a_i} \right\|_{A_p} \leqslant k.$$

Hence $\sup_n \|f_1^n\|_A < \infty$. If $f \in A_p(G)$ is arbitrary with $|f(x)| \leqslant 1$, let $f_1(x) = f(x)1_F(x)$ where $F = \{x;\ |f(x)| = 1\}$ and $f_2 = f(1 - 1_F)$. Then $\sup_x |f_2(x)| = \alpha < 1$, so $\|f_2^n\|_{A_p}^{1/n} \to \alpha < 1$, i.e. $\|f_2^n\|_{A_p} \to 0$. Now

$$\|f^n\|_{A_p} = \|f_1^n + f_2^n\| \leqslant \|f_1^n\| + \|f_2^n\|.$$

Since $\|f_1^n\|$ is bounded by the first part, if follows that $\sup_n \|f^n\| < \infty$.

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