

ON THE SYMMETRY OF MATRIX ALGEBRAS

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ABSTRACT. A $*$ -algebra is called symmetric, if each element of the form a^*a has nonnegative real spectrum. The study of locally compact groups with symmetric group algebras led to the following theorem: The tensoring of a Banach $*$ -algebra with the $*$ -algebra of all complex $n \times n$ matrices preserves symmetry. In this note we prove, by a very simple algebraic argument, an analogue of it for arbitrary $*$ -algebras.

1. Introduction. The problem of determining which locally compact groups have symmetric group algebras has received much attention. It often led to the study of the question whether the tensor product of two symmetric Banach $*$ -algebras A and B is again symmetric [2], [1], [4], [5].

In the special case where B is the finite dimensional $*$ -algebra C_n of all complex $n \times n$ matrices the tensor product of A and B is the $*$ -algebra A_n of all $n \times n$ matrices over A with the usual algebraic operations and involution $(a_{ij})^* = (a_{ji}^*)$. It was proved by H. Leptin [6] and the author [7] that if A is a symmetric Banach $*$ -algebra, then the matrix algebra A_n is also symmetric. (As pointed out in [6] D. W. Bailey's proof [1] of the 2×2 case is incorrect.)

The purpose of this note is to prove, by a very simple algebraic argument, the following more general result.

THEOREM. *Let A be a $*$ -algebra and let n be a positive integer. Then A has the property that each element of the form $a_1^*a_1 + \cdots + a_k^*a_k$, $k = 1, 2, 3, \dots$, has nonnegative real spectrum if and only if the $*$ -algebra A_n of all $n \times n$ matrices over A has the same property.*

This answers a question of R. S. Doran [3].

2. Proof of the Theorem. Let A be an arbitrary $*$ -algebra and let k and n be positive integers. For convenience we say that A is k -symmetric, if for every set of k elements a_1, \dots, a_k in A the element $a_1^*a_1 + \cdots + a_k^*a_k$ has nonnegative real spectrum. Clearly, for any $*$ -algebra, k -symmetric implies symmetric, i.e. 1-symmetric. It is well known that for Banach $*$ -algebras the converse holds.

2.1. LEMMA. *If A is k -symmetric, then the $*$ -algebra A_e obtained from A by adjunction of an identity element e is also k -symmetric.*

If the $$ -algebra $(A_e)_n$ of all $n \times n$ matrices over A_e is k -symmetric, then the $*$ -algebra A_n of all $n \times n$ matrices over A is also k -symmetric.*

PROOF. Given a k -tuple $\lambda = (\lambda_1, \dots, \lambda_k)$ of complex scalars and a set of k

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elements a_1, \dots, a_k in A , set

$$h = \sum (\lambda_r e + a_r)^* (\lambda_r e + a_r) - \sum |\lambda_r|^2 e \quad \text{and} \quad |\lambda|^2 = \sum |\lambda_r|^2$$

(sums are taken over $r = 1, \dots, k$). Since h is a Hermitian element in A we can decompose the following element into commuting factors.

$$\begin{aligned} e + \sum [(\lambda_r e + a_r)(1 + |\lambda|^2)^{-1} h]^* [(\lambda_r e + a_r)(1 + |\lambda|^2)^{-1} h] \\ = e + \sum h(1 + |\lambda|^2)^{-1} (\lambda_r e + a_r)^* (\lambda_r e + a_r)(1 + |\lambda|^2)^{-1} h \\ = e + h(1 + |\lambda|^2)^{-1} (h + |\lambda|^2 e)(1 + |\lambda|^2)^{-1} h \\ = (e + h(1 + |\lambda|^2)^{-1} (h - e))(1 + |\lambda|^2)^{-1} (e + |\lambda|^2 e + h). \end{aligned}$$

Thus, if A is k -symmetric, then

$$e + |\lambda|^2 e + h = e + \sum (\lambda_r e + a_r)^* (\lambda_r e + a_r)$$

is invertible in A_e , i.e. A_e is k -symmetric (see also [7]).

The assertion about A_n follows from the fact that A_n is a $*$ -ideal in $(A_e)_n$.

The preceding lemma shows that we may assume without loss of generality that A has an identity element e . We denote the identity matrix in A_n by e_n .

2.2. REMARK. If A_n is k -symmetric, then A is $(n \cdot k)$ -symmetric. Indeed, if A_n is k -symmetric, then for the $n \times n$ matrices $a_r = (a_{rij})$, with $a_{rij} = 0$ for $j > 1$, the $n \times n$ matrix

$$e_n + \sum a_r^* a_r = \begin{pmatrix} e + \sum (a_{r11}^* a_{r11} + \dots + a_{rn1}^* a_{rn1}) & 0 \\ 0 & e_{n-1} \end{pmatrix}$$

is invertible in A_n . Thus $e + \sum (a_{r11}^* a_{r11} + \dots + a_{rn1}^* a_{rn1})$ is invertible in A , i.e. A is $(n \cdot k)$ -symmetric. This proves one direction of the theorem.

2.3. LEMMA. If A is $(2k + 1)$ -symmetric, then the $*$ -algebra A_2 is k -symmetric.

PROOF. Given k matrices $a_r = (a_{rij})$, $r = 1, \dots, k$, in A_2 , we have to show that the matrix

$$\begin{aligned} e_2 + \sum a_r^* a_r \\ = \begin{pmatrix} e + \sum (a_{r11}^* a_{r11} + a_{r21}^* a_{r21}) & b \\ b^* & e + \sum (a_{r12}^* a_{r12} + a_{r22}^* a_{r22}) \end{pmatrix} \end{aligned}$$

is invertible in A_2 . Consider the invertible 2×2 matrix

$$x = \begin{pmatrix} e & -(e + \sum (a_{r11}^* a_{r11} + a_{r21}^* a_{r21}))^{-1} b \\ 0 & e \end{pmatrix},$$

and set $y_r = a_r x$. Simple matrix multiplication shows that

$$\begin{aligned} x^* (e_2 + \sum a_r^* a_r) x &= x^* x + \sum y_r^* y_r \\ &= \begin{pmatrix} e + \sum (a_{r11}^* a_{r11} + a_{r21}^* a_{r21}) & 0 \\ 0 & e + x_{12}^* x_{12} + \sum (y_{r12}^* y_{r12} + y_{r22}^* y_{r22}) \end{pmatrix} \end{aligned}$$

is also invertible. Thus $e_2 + \sum a_r^* a_r$ is invertible in A_2 .

2.4. PROOF OF THE THEOREM. We have shown that the theorem holds for $n = 2^1$. If we assume the theorem for $n = 2^m$, then it follows for $n = 2^{(m+1)}$ from the fact that the $*$ -algebra of $2^{(m+1)} \times 2^{(m+1)}$ matrices over A is $*$ -isomorphic to the $*$ -algebra of 2×2 matrices over A_{2^n} by partitioning. Thus the theorem holds for all $n = 2^m$.

Now let n be an arbitrary positive integer. Choose a positive integer m such that $n < 2^m$. Then A_n is $*$ -isomorphic to the $*$ -subalgebra of all $2^m \times 2^m$ matrices over A of the form

$$\begin{pmatrix} a & 0 \\ 0 & e_{(2^m-n)} \end{pmatrix}, \quad \text{where } a \text{ is in } A_n.$$

The inverse of a matrix of this form, if it exists in A_{2^m} , is of the form

$$\begin{pmatrix} a' & 0 \\ 0 & e_{(2^m-n)} \end{pmatrix}, \quad \text{where } a' \text{ is in } A_n \text{ and } a'a = e_n = aa'.$$

Thus, if A_{2^m} is k -symmetric, then also A_n is k -symmetric. This proves the theorem for every positive integer n .

2.5. EXAMPLE. The field $C(x_1, \dots, x_n)$ of fractions of complex polynomials in x_1, \dots, x_n , with involution defined by $x_i^* = x_i, i = 1, 2, \dots, n$, is an example of a nonnormable $*$ -algebra which is k -symmetric for every positive integer k .

3. Extensions. Let A be a $*$ -algebra and I a $*$ -ideal in A . Clearly, if A is symmetric, then also I and A/I are symmetric. The converse is still an open problem (even for Banach $*$ -algebras). Here are some observations concerning this problem.

3.1. Observe that A is a $*$ -ideal in A_e such that the quotient algebra A_e/A , which is $*$ -isomorphic to the Banach $*$ -algebra C of complex numbers, is symmetric. Thus our Lemma 2.1 is a special case of this problem.

3.2. The radical R of a $*$ -algebra A is a $*$ -ideal in A and every radical $*$ -algebra is symmetric. Furthermore, A is symmetric if and only if A/R is symmetric.

3.3. It is well known that a Banach $*$ -algebra is symmetric if and only if it is Hermitian, i.e. every Hermitian element has real spectrum.

Now let A be commutative and assume that I and A/I are Hermitian. We claim that then also A is Hermitian. Indeed, if h is a Hermitian element in A , then, since A/I is Hermitian, $(e + ih)x = e + a + ib$, for some x in A , and Hermitian a and b in I . Thus $(e + ih)(x - a) = e + i(b - ha)$, where $b - ha$ is a Hermitian element in I . Since I is Hermitian, $e + i(b - ha)$ is invertible, and hence also $e + ih$. This shows that the spectrum of h in A is real.

3.4. A Banach $*$ -algebra is called a B^* -algebra, if $\|a^*a\| = \|a\|^2$ for all elements a . It is well known that every B^* -algebra is symmetric.

Now assume that I and A/I are B^* -algebras. Then also A is a B^* -algebra with respect to the norm

$$\|a\| = \max\{\|a + I\|, \sup\{\|ax\|: x \text{ in } I \text{ and } \|x\| \leq 1\}\}.$$

4. Completion. Let A be a normed algebra with continuous involution. Then the completion \bar{A} of A is a Banach $*$ -algebra. We do not know if the completion \bar{A} is Hermitian whenever A is Hermitian. Here are some remarks concerning this problem.

4.1. For any $*$ -algebra A consider the $*$ -algebra A_0 of all infinite matrices $a = (a_{ij})$ with $a_{ij} = 0$ for almost all i and j . If A is k -symmetric for all $k = 1, 2, 3, \dots$, then A_0 has the same property.

In the special case where A is a symmetric Banach algebra with continuous involution, the completion of A_0 with respect to the norm $\|a\| = \sum \|a_{ij}\|$ is again symmetric (H. Leptin, private communication).

4.2. In a commutative Banach algebra the spectrum is continuous. Thus, if A is a commutative Hermitian normed algebra with continuous involution, then the completion of A is again Hermitian.

4.3. The product $\prod A_i$ of a family of $*$ -algebras A_i is a $*$ -algebra with respect to the involution $(a_i)^* = (a_i^*)$. The quasi-inverse of a quasi-regular element (a_i) in $\prod A_i$ is given by $(-a_i(e - a_i)^{-1})$ (with $\|a_i(e - a_i)^{-1}\| \leq \|a_i\|(1 - \|a_i\|)^{-1}$ if A_i is normed and $\|a_i\| < 1$). Hence the product $\prod A_i$ of a family of $*$ -algebras A_i is symmetric if and only if each A_i is symmetric.

A similar remark holds for the $*$ -ideal $\sum A_i$ of all (a_i) in $\prod A_i$ with $a_i = 0$ for almost all i , for the normed $*$ -algebra of all (a_i) in $\prod A_i$ with $\|(a_i)\| = \sum \|a_i\| < \infty$, and for the normed $*$ -algebra of all (a_i) in $\prod A_i$ with $(\|a_i\|)$ vanishing at infinity and norm $\|(a_i)\| = \sup \|a_i\|$, when the $*$ -algebras A_i are normed.

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