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THE DEVIL AND THE ANGEL OF LOOPS

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ABSTRACT. G_a, the subloop generated by all the associators of a loop G, is singled out for study for the first time. If G is Moufang, G_a is found to be normal in G. The relation of G_a with the nucleus of G is also investigated.

A binary system $(G, \cdot)$ is a loop if (i) $(G, \cdot)$ is closed, (ii) $(G, \cdot)$ has an identity 1, (iii) $x, y \in G \Rightarrow$ there exist unique $u, v \in G$ such that $xu = y$, $vx = y$.

A group is a loop; but a loop may not be a group. The difference lies on the Associative Law.

For $x, y, z \in G$, we can write $xy \cdot z = (x \cdot yz)(x,y,z)$.$(x,y,z)$ is called the associator of x, y, z. The subloop G_a, generated by all the associators of G, is called the associator subloop of G. If $G_a = 1$, then G is obviously a group and everything will be fine. On the other hand, if G_a is nontrivial, the loop may be so difficult that even the best genius will fight shy of it. It is therefore not inappropriate to call G_a the devil of G.

In contrast with this, we have the nucleus N of G. $N \subset G$ and for any $n \in N$, $(n,x,y) = (x,n,y) = (x,y,n) = 1$ for all $x, y \in G$. Clearly N is a group. It helps us as a stepping stone to understand the loop G. It is therefore not inappropriate to call N the angel of G.

For an arbitrary loop G, nothing much can be said about the devil and the angel. If G is Moufang, i.e., $xy \cdot zx = (x \cdot yz)x$ for all $x, y, z \in G$, then they become beautiful:

THEOREM 1. *If G is a Moufang loop, then $N \lhd G$.*

PROOF. By [1, p. 114, Theorem 2.1] and by disassociativity of G.

We wish to investigate if the devil also possesses this property:

DEFINITION. Let $(G, \cdot)$ be a loop. $A(x,y,z)$ and $B(x,y,z)$ are defined as:

(a) $(xy)z = [A(x,y,z) \cdot x](yz)$;

(b) $x(yz) = [(B(x,y,z) \cdot x) \cdot y] \cdot z$ for all $x, y, z \in G$.

THEOREM 2. *Let $(G, \cdot)$ be a loop; $H = \langle A(x,y,z), B(x,y,z) | x,y,z \in G\rangle$. Then* (i) $(Hx)y = H(xy)$, $H[(xy)z] = H[x(yz)]$ *for all* $x, y, z \in G$, (ii) *H is the smallest normal subloop of G such that G / H is a group.*

PROOF. (i) By (a), with $x = h \in H$, $(hy)z \in H(yz)$ for all $h \in H$. Therefore $(Hy)z \subset H(yz)$ for all $y, z \in G$. By (b), taking $x = h \in H$, $h(yz) \in (Hy) \cdot z$ for all $h \in H$. Therefore $H(yz) \subset (Hy) \cdot z$ for all $y, z \in G$. So $(Hx)y = H(xy)$ for all $x, y \in G$.

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By (a), $xy \cdot z \in (Hx)(yz)$. But $(Hx)(yz) = H(x \cdot yz)$. Hence $(xy)z = h_1 \cdot x(yz)$ for some $h_1 \in H$. Then

$$H((xy)z) = H[h_1 \cdot (x \cdot yz)] = (Hh_1) \cdot (x \cdot yz) = H \cdot (x \cdot yz).$$

(ii) For each $x \in G$, define the map α_x by $(Hy)\alpha_x = (Hy) \cdot x$ for all $y \in G$. It is easy to verify that α_x is a permutation of the set of right cosets Hy, $y \in G$.

For each $x \in G$, define α by $x\alpha = \alpha_x$. Then for all $x, y, z \in G$,

$$\begin{aligned}(Hx)\alpha_y\alpha_z &= [(Hx)y] \cdot \alpha_z = (H(xy)) \cdot z = H(xy \cdot z) \\ &= H(x(yz)) = (Hx)\alpha_{yz}.\end{aligned}$$

Therefore $\alpha_y\alpha_z = \alpha_{yz}$ or $(yz)\alpha = y\alpha z\alpha$ for all $y, z \in G$. So α is a homomorphism of $(G, \cdot)$ into a permutation group. $x \in \ker(\alpha) \Leftrightarrow (Hy)x = Hy$ for all $y \in G$. Taking $y \in H$, we have $Hx = H$ or $x \in H$. So $\ker(\alpha) \subset H$. As $G/\ker(\alpha)$ is a group, it can be seen easily that $\ker(\alpha)$ contains $A(x,y,z)$ and $B(x,y,z)$ for all $x, y, z \in G$. So $\ker(\alpha) \supset H$. Hence, $H = \ker(\alpha)$.

Let S be another normal subloop of G such that G / S is a group. Then

$$(xS \cdot yS)zS = [A(x,y,z)S \cdot xS] \cdot [yS \cdot zS].$$

By cancellation, we have $A(x,y,z) \in S$. Similarly $B(x,y,z) \in S$. Thus $H \subset S$.

COROLLARY. *Let G be a Moufang loop. Then G_a is the smallest normal subloop of G such that G/G_a is a group.*

PROOF. Let $x, y, z \in G$. By [1, p. 124, Lemma 5.4 (5.13) and (5.16)], $zR(x,y) = zL(x^{-1},y^{-1}) = z(z,y^{-1},x^{-1})^{-1}$. Similarly, we have $z^{-1}R(x,y) = z^{-1}(z^{-1},y^{-1},x^{-1})^{-1}$. As $R(x,y)$ is a pseudoautomorphism of G, $z^{-1}R(x,y) = (zR(x,y))^{-1}$. Thus $zR(x,y) = (z^{-1},y^{-1},x^{-1}) \cdot z$.

By [1, p. 124, Lemma 5.4, (5.13) and (5.16)], $zR(x,y)^{-1} = zL(y^{-1},x^{-1}) = z(z,x^{-1},y^{-1})^{-1}$. Thus $z^{-1}R(x,y)^{-1} = z^{-1}(z^{-1},x^{-1},y^{-1})^{-1}$. But $z^{-1}R(x,y)^{-1} = [zR(x,y)^{-1}]^{-1}$. Therefore $zR(x,y)^{-1} = (z^{-1},x^{-1},y^{-1})z$.

By definition of $A(x,y,z)$ and $B(x,y,z)$, $zR(x,y) = A(z,x,y) \cdot z$ and $zR(x,y)^{-1} = B(z,x,y) \cdot z$. Therefore $A(z,x,y) = (z^{-1},y^{-1},x^{-1})$, $B(z,x,y) = (z^{-1},x^{-1},y^{-1})$. So

$$\begin{aligned}H &= \langle A(z,x,y), B(z,x,y) | x,y,z \in G\rangle \\ &= \langle (z^{-1},y^{-1},x^{-1}), (z^{-1},x^{-1},y^{-1}) | x,y,z \in G\rangle = G_a.\end{aligned}$$

THEOREM 3. *Let G be a loop and H a normal subloop of G such that $H \leqq N$. Then*:

(a) $G/C_G(H) \leqq \operatorname{Aut} H$ *where* $C_G(H) = \{g | g \in G, hg = gh,$ *for all* $h \in H\}$.

(b) $C_G(H) \cap H = Z(H)$, *the center of* H.

PROOF. Let $x \in G$. Define T_x by means of $hx = x \cdot hT_x$ for all $h \in H$. By normality $Hx = xH$. Thus, T_x is a permutation of H. If $h, h' \in H$, then,

$$\begin{aligned}
x((hh')T_x) &= (hh')x && \text{by definition,}\\
&= h(h'x) && \text{as } h' \in H \subset N,\\
&= h(x(h'T_x)) && \text{by definition,}\\
&= hx \cdot (h'T_x) && \text{since } h'T_x \in H \subset N,\\
&= (x \cdot hT_x)(h'T_x) && \text{by definition,}\\
&= x(hT_x \cdot h'T_x) && \text{as } h'T_x \in H \subset N.
\end{aligned}$$

Therefore, $(hh')T_x = hT_x \cdot h'T_x$ for all $h, h' \in H$, for all $x \in G$. T_x is an automorphism of H for each $x \in G$.

Consider the map $x \to T_x$, $x \in G$. Let $y \in G$. Then for all $h \in H$,

$$\begin{aligned}
(xy)(hT(xy)) &= h(xy) && \text{by definition,}\\
&= hx \cdot y && \text{as } H \leqq N,\\
&= x(hT_x) \cdot y && \text{by definition,}\\
&= x \cdot hT_x y && \text{as } hT_x \in H \subset N,\\
&= x(y(hT_x T_y)) && \text{by definition,}\\
&= (xy)(hT_x T_y) && \text{as } hT_x T_y \in N.
\end{aligned}$$

Thus $hT_{xy} = hT_x T_y$ and $T_{xy} = T_x T_y$ for all $x, y \in G$. So, α is a homomorphism of G into $\mathrm{Aut}(H)$.

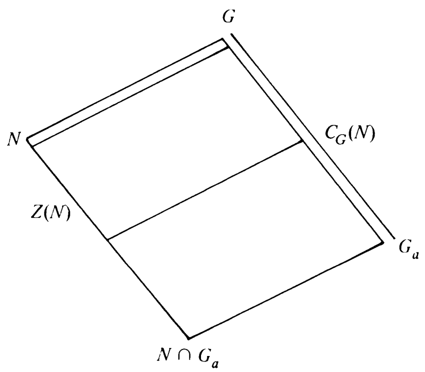
Let $x \in G$. $T_x = I \Leftrightarrow hx = xh$ for all $h \in H$. Therefore, the kernel is $C_G(H)$ and (a) is proven.

Since $H \subset N$, H is a group. So (b) is clear.

COROLLARY. *Let G be a Moufang loop. Then $G_a \subset C_G(N)$.*

PROOF. Let $H = N$ in the Theorem. As $G/C_G(N)$ is a group, $G_a \subset C_G(N)$.

The relations between the angel and the devil in the Moufang case are as shown in the diagram:



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