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INDUCED AUTOMORPHISMS AND SIMPLE APPROXIMATIONS

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ABSTRACT. A class of ergodic, measure preserving invertible point transformations, which are said to admit simple approximations is defined below. If T is an automorphism which admits a simple approximation, conditions are given on a set A so that the induced automorphisms T^A and T_A again admit simple approximations.

1. **Preliminaries.** Let (X, F, μ) be a measure space isomorphic to the unit interval with Lebesgue measure. A measure preserving invertible point transformation of X is called an *automorphism* of (X, F, μ).

DEFINITION 1. A finite ordered collection $\xi = \{A_i\colon 1 \leqq i \leqq m\}$ of pairwise disjoint measurable sets in X is called a *partition.* If the union of members of ξ is X, then ξ is called a *partition of* X. If $A \in F$ we write $A \leqq \xi$ if A is a union of members of ξ. If $\eta = \{B_j\colon 1 \leqq j \leqq n\}$ is a partition, we write $\eta \leqq \xi$ if $B_j \leqq \xi$ for $j = 1, \ldots, n$.

DEFINITION 2. Let ε denote the partition of X into single points. We shall say that a sequence of partitions $\{\xi(n)\}$ converges to the *unit partition*, and we write $\xi(n) \to \varepsilon$ if for each $A \in F$, $\mu(A \triangle A(\xi(n))) \to 0$ as $n \to \infty$, where $A(\xi(n)) \leqq \xi(n)$ and is such that $\mu(A \triangle A(\xi(n)))$ is a minimum.

Following, we define the class of automorphisms that admit simple approximation.

DEFINITION 3. An automorphism T is said to admit a *simple approximation* if there exists a sequence of partitions $\{\xi(n)\}$, $\xi(n) = \{C_i(n)\colon i = 1, \ldots, q(n)\}$ with the property that

(i) $\xi(n) \to \varepsilon$ as $n \to \infty$,

(ii) $TC_i(n) = C_{i+1}(n)$ for $i = 1, \ldots, q(n) - 1$.

Chacon and Schwartzbauer [2] require the additional condition that $\lim_{n\to\infty} q(n)\mu(X \setminus \bigcup_{i=1}^{q(n)} C_i(n)) = 0$. This condition will not be required by us, but we shall see that a similar condition arises naturally in the discussion of the induced automorphisms T^A and T_A. In fact, Schwartzbauer [5] has shown that if $\lim_{n\to\infty} q(n)\mu(X \setminus \bigcup_{i=1}^{q(n)} C_i(n)) = c < \infty$, then T cannot be strongly mixing.

It is well known that automorphisms that admit simple approximation are ergodic and have simple spectrum [5], [2].

2. **The induced automorphisms T^A and T_A.** Let $T\colon X \to X$ be an automorphism and $A \in F$ a set with positive measure.

DEFINITION 4. Let A' be a copy of A, $\tau\colon A \to A'$ a one-to-one map, and $X^A = X \cup A'$. Then the *primitive transformation* $T^A\colon X^A \to X^A$ is defined by

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$$T^A(x) = \begin{cases} \tau(x), & x \in A, \\ T(x), & x \in X \setminus A, \\ T(\tau^{-1}(x)), & x \in A'. \end{cases}$$

DEFINITION 5. We define the *derivative transformation* $T_A: A \to A$ by

$$T_A(x) = T^n(x), \qquad x \in A,$$

where n is the least integer such that $T^n(x) \in A$ (neglecting sets of measure zero).

Both T^A and T_A are called *induced transformations*. When X^A and A are made into probability spaces in the obvious way, then T^A and T_A become automorphisms which are ergodic if and only if T is ergodic.

Kakutani [3] first introduced the idea of induced transformation, and in [4] he gave an example of an induced automorphism which is weakly mixing but not strongly mixing. In this example the underlying automorphism T admits a simple approximation.

DEFINITION 6. We can define a metric ρ on the set of ordered partitions with m elements (neglecting sets of measure zero).

If $\xi = \{A_i: i = 1, \ldots, m\}$, $\eta = \{B_j: j = 1, \ldots, m\}$ put

$$\rho(\xi, \eta) = \sum_{i=1}^{m} \mu(A_i \triangle B_i).$$

The measure algebra (F, μ) is a complete metric space with respect to the metric d given by

$$d(A, B) = \mu(A \triangle B), \qquad A, B \in F.$$

We need shall the following lemma (Baxter [1]).

LEMMA 1. *Let $\xi(n) = \{A_i(n): i = 1, \ldots, q(n)\}$, $\eta(n) = \{B_j(n): j = 1, \ldots, q(n)\}$ be sequences of partitions such that $\xi(n) \to \varepsilon$ and $\rho(\xi(n), \eta(n)) \to 0$, then $\eta(n) \to \varepsilon$.*

3. **Main theorems.** We shall prove the results for the primitive automorphism T^A and will outline the proofs for the derived automorphism T_A.

THEOREM 1. *Let $T: X \to X$ be an automorphism which admits a simple approximation; then there is a set of subsets of X, dense in F, such that the induced automorphisms T^A and T_A on any one of these sets also admit a simple approximation.*

PROOF. By a result of Baxter [1] we may assume that T admits a simple approximation with respect to an increasing sequence of partitions $\xi(n)$, i.e. $\xi(n) \leqq \xi(n+1)$ for all n.

Let $\xi(n) = \{C_i(n): i = 1, \ldots, q(n)\}$ and fix $m \geqq 1$. If we put $A = C_j(m)$ for some j, $1 \leqq j \leqq q(m)$, then it is easy to see that T^A again admits a simple approximation, and so the result follows. The proof for T_A is similar.

LEMMA 2. *Let $T: X \to X$ be an automorphism. If A is a measurable set with* *positive measure which can be approximated by a sequence of measurable sets $A(n) \subset A$ in the sense that $\mu(A \backslash A(n)) \to 0$ as $n \to \infty$, then the sequence of transformations $\{T_n\}$ defined by*

$$T_n(x) = \begin{cases} T^{A(n)}(x), & x \in A'(n) \cup X, \\ x, & x \in A' \backslash A'(n), \end{cases}$$

converges to T^A in the uniform topology. ($T^{A(n)}$ is the primitive automorphism induced by T on $A(n)$. $A'(n)$ and A' are copies of $A(n)$ and A respectively.)

PROOF. Clearly the automorphisms T^A and T_n coincide on the sets $A(n)$, $X \backslash A$ and $A'(n)$, so they can only differ on the sets $A \backslash A(n)$, $A' \backslash A'(n)$. Therefore

$$\mu\{x\colon T_n(x) \neq T^A(x)\} \leqq \mu[(A \backslash A(n)) \cup (A' \backslash A'(n))]$$
$$\leqq 2\mu(A \backslash A(n)) \to 0 \quad \text{as } n \to \infty.$$

Hence $T_n \to T^A$ in the uniform topology.

REMARK. The corresponding result for the derived automorphism T_A is true provided we assume, in addition, that the automorphism T admits a simple approximation and that $A(n) \leqq \xi(n)$.

Following is our main theorem.

THEOREM 2. *Let T admit a simple approximation with respect to a sequence of partitions $\{\xi(n)\}$, $\xi(n)$ having $q(n)$ elements, and suppose $A \in F$ with $\mu(A) > 0$ can be approximated by sets $A(n) \subset A$ with $A(n) \leqq \xi(n)$ and such that $q(n)\mu(A \backslash A(n)) \to 0$ as $n \to \infty$. Then T^A and T_A, the induced automorphisms on A, admit a simple approximation.*

REMARK. We prove the theorem for the primitive automorphism T^A. The proof for the derived automorphism T_A is similar.

PROOF. $A(n) \leqq \xi(n)$, so assume that $A(n)$ is the union of $p(n)$ elements of $\xi(n)$, $n = 1, 2, \ldots . A'(n) \subset A'$, so we can construct a sequence of partitions for $X \cup A'$ consisting of the $q(n)$ elements of $\xi(n)$ together with the $p(n)$ elements of $A'(n)$ (which are just copies of the $\xi(n)$-*sets* of $A(n)$). Denote this partition by $\beta(n)$ and give it the natural order obtained from the transformation $T^{A(n)}$.

Put $\beta(n) = \{D_i(n)\colon i = 1, \ldots, p(n) + q(n)\}$. Clearly, as $n \to \infty \beta(n) \to \varepsilon^A$, the point partition of $X \cup A'$, and also

$$D_i(n) = T_n^{i-1} D_1(n) \quad \text{for } i = 1, \ldots, p(n) + q(n),$$

where T_n is the automorphism defined in Lemma 2.

Define a second sequence of partitions for $X \cup A'$, denoted by $\{\eta(n)\}$ where

$$\eta(n) = \{E_i(n)\colon i = 1, \ldots, p(n) + q(n)\}$$

and

$$E_1(n) = D_1(n),\ E_i(n) = (T^A)^{i-1} D_1(n), \qquad i = 1, \ldots, p(n) + q(n).$$

We shall show that T^A admits a simple approximation with respect to $\eta(n)$. It suffices to show that $\eta(n) \to \varepsilon^A$ as $n \to \infty$. We show that $\rho(\beta(n), \eta(n)) \to 0$, and since $\beta(n) \to \varepsilon^A$, the result will follow from Lemma 1.

$$\begin{aligned}\rho(\beta(n), \eta(n)) &= \sum_{i=1}^{p(n)+q(n)} \mu(D_i(n) \triangle E_i(n)) \\ &= \sum_{i=0}^{p(n)+q(n)-1} \mu(T_n^i D_1(n) \triangle (T^A)^i D_1(n)).\end{aligned}$$

But T_n approximates T^A in the uniform topology. In fact if $x \in D_1(n)$ then

$$T_n^i(x) = (T^A)^i(x)$$

unless

$$x \in \bigcup_{l=0}^{i-1} (T^A)^{-l}[(A \backslash A(n)) \cup (A' \backslash A'(n))],$$

i.e. unless $x \in \bigcup_{l=0}^{i-1} (T^A)^i(G(n))$ where $G(n) = (A \backslash A(n)) \cup (A' \backslash A'(n))$. It follows that

$$\begin{aligned}\tfrac{1}{2}\mu(T_n^i D_1(n) \triangle (T^A)^i D_1(n)) &\leqq \mu\left[\left(\bigcup_{l=0}^{i-1} (T^A)^{-l} G(n)\right) \cap D_1(n)\right] \\ &\leqq \mu\left[\bigcup_{l=0}^{p(n)+q(n)-1} (T^A)^{-l} G(n) \cap D_1(n)\right] \\ &\leqq \sum_{l=0}^{p(n)+q(n)-1} \mu[(T^A)^{-l} G(n) \cap D_1(n)] \\ &= \sum_{l=0}^{p(n)+q(n)-1} \mu[G(n) \cap (T^A)^l D_1(n)] \\ &= \mu\left[G(n) \cap \bigcup_{l=0}^{p(n)+q(n)-1} (T^A)^l D_1(n)\right] \\ &\leqq \mu(G(n)) \leqq 2\mu(A \backslash A(n)).\end{aligned}$$

Hence

$$\begin{aligned}\tfrac{1}{2}\rho(\beta(n), \eta(n)) &\leqq 2 \sum_{i=0}^{p(n)+q(n)-1} \mu(A \backslash A(n)) \\ &= 2(p(n) + q(n))\mu(A \backslash A(n)) \\ &\leqq 4q(n)\mu(A \backslash A(n)) \\ &\to 0 \quad \text{as } n \to \infty.\end{aligned}$$

Thus $\eta(n) \to \varepsilon^A$ and so T^A admits a simple approximation.

Recall that if T admits a simple approximation with respect to a sequence of partitions $\xi(n) = \{C_i(n)\colon i = 1, \ldots, q(n)\}$ with the property that

$q(n)\mu(X \setminus \bigcup_{i=1}^{q(n)} C_i(n)) \to c < \infty$, then T is not strongly mixing. From this we deduce

COROLLARY 1. *Suppose that T and $A \in F$ satisfy the hypothesis of Theorem* 2; *then the derived automorphism $T_A: A \to A$ is not strongly mixing.*

REMARK. We cannot deduce that the primitive automorphism T^A is also not strongly mixing.

BIBLIOGRAPHY

1. J. R. Baxter, *A class of ergodic transformations having simple spectrum*, Proc. Amer. Math. Soc. **27** (1971), 275–279. MR **43** #2187.

2. R. V. Chacon and T. Schwartzbauer, *Commuting point transformations*, Z. Wahrscheinlichkeitstheorie und Verw. Gebiete **11** (1969), 277–287. MR **39** #2939.

3. S. Kakutani, *Induced measure preserving transformations*, Proc. Imp. Acad. Tokyo **19** (1943), 635–641. MR **7**, 255.

4. ——, *Examples of ergodic measure preserving transformations which are weakly mixing but not strongly mixing*, Lecture Notes in Math., vol. 318, Springer-Verlag, Berlin and New York, 1973, pp. 143–149.

5. T. Schwartzbauer, *Automorphisms that admit of approximations by periodic transformations*, Ph.D. Thesis, University of Minnesota, 1968.

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