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ON THE BEHAVIOR OF MEROMORPHIC FUNCTIONS AT THE IDEAL BOUNDARY OF A RIEMANN SURFACE

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ABSTRACT. In a former work the author established an analog of a classical theorem of Painlevé in the context of an arbitrary resolutive compactification of a Riemann surface. In the same setting, a refinement of the argument used in the above yields an elementary proof of a theorem of Riesz-Luzin-Privaloff type: If a meromorphic function f tends to zero at each point of a subset E of the ideal boundary and E has positive harmonic measure, then $f \equiv 0$ on R. The well-known inclusion relations $U_{HB} \subset \mathcal{O}_{MB^*}$, and $U_{HD^-} \subset \mathcal{O}_{MD^*}$, are then established from the point of view of the resolutivity of the Wiener and Royden compactification respectively.

1. **Introduction.** The boundary behavior of analytic and meromorphic functions has been a subject of investigation by mathematicians for more than three-quarters of a century. Early classical developments were due to Painlevé, Fatou, F. and M. Riesz, Luzin-Privaloff, Plessner, and Nevanlinna, with the theory being subsequently developed by numerous other authors. The unit disk, $|z| < 1$, has been the domain of consideration in most of these works.

The ideal boundary behavior of a meromorphic function on a Riemann surface, which is the subject of this note, has been studied by Constantinescu and Cornea [1], and Sario and Nakai [4], amongst others. In the next section, we give a proof of a theorem of Riesz-Luzin-Privaloff type for an arbitrary resolutive compactification of a Riemann surface. The method of proof is both elemental in nature and fundamentally different from those given in the two aforementioned works. The Theorem is then used to establish a single direct line of proof of the well-known inclusion relations $U_{HB} \subset \mathcal{O}_{MB^*}$, and $U_{HD^-} \subset \mathcal{O}_{MD^*}$.

2. **Preliminaries.** Let R^* be a resolutive compactification of a Riemann surface R, and set $\Delta = R^* - R$. The harmonic measure on Δ we denote by $\omega^{R^*} = \omega$. We quote the following result [2, Hilfssatz 8.8] as it is instrumental to our proof of the Main Theorem.

LEMMA. *Let R^* be a resolutive compactification and G an open subset of R. If s is a positive superharmonic function on G, then the set*

$$(1) \qquad A = \left\{ \zeta \in \Delta - \overline{R - G} : \varliminf_{R \ni z \to \zeta} s(z) = \infty \right\}$$

has harmonic measure zero.

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3. **Main results.** We present here an elementary proof of a theorem of Riesz-Luzin-Privaloff type (cf. Constantinescu and Cornea [1], Sario and Nakai [4], Schiff [5]).

THEOREM. *Let R^* be a resolutive compactification of a Riemann surface R, and f a meromorphic function on R such that $\lim_{R \ni z \to \zeta} f(z) = 0$, for all $\zeta \in E \subset \Delta$. If E has positive harmonic measure, then $f \equiv 0$ on R.*

PROOF. Suppose $f \not\equiv 0$ on R and let $R' = R - f^{-1}(\infty)$. Then R' is open in R, and the function $s = -\log|f|$ is superharmonic on R'. Hence the set $G = \{z \in R' : s(z) > 0\}$ is an open subset of R', and a fortiori G is open in R. The hypotheses of the Lemma are now satisfied, so that $\omega(A) = 0$, where A is defined in (1). For any $\zeta \in E \subset \Delta$, if $\zeta \notin A$, then $\lim_{R \ni z \to \zeta} s(z) = \infty$ implies $\zeta \in \overline{R - G}$. Hence there exists a net $\{z_\alpha\} \subset R - G$ such that $z_\alpha \to \zeta$. However, $s(z_\alpha) \leqslant 0$ (even if $z_\alpha \in f^{-1}(\infty)$), which violates the fact that $\lim_{z_\alpha \to \zeta} s(z_\alpha) = \infty$. It follows that $E \subset A$ and $\omega(E) = 0$, contradicting $\omega(E) > 0$. We conclude that $f \equiv 0$ on R.

From the proof of the Theorem we obtain the following result:

COROLLARY 1. *If there exists a superharmonic function s on R such that $s \to \infty$ at each point of $E \subset \Delta$, then $\omega(E) = 0$.*

Whether or not the converse of this result is valid is an open question, however, it is well known to be false if we require s to be positive.

In passing, we remark that if $p \in \Delta$ with $\omega(p) > 0$, then it follows from the Theorem that any meromorphic function f on R with a continuous extension to p satisfies $f(p) \neq \pm\infty$.

Let U_X denote the class of Riemann surfaces which carry at least one X-minimal function for $X = HB, HD^{\sim}$. Here, HB represents the class of bounded harmonic functions, and $HD^{\sim}$ the class of limits of nonincreasing sequences of positive Dirichlet-finite harmonic functions on R which converge uniformly on compact subsets of R. Moreover, denote by $\mathcal{O}_X$ the null classes of Riemann surfaces for $X = MB^*$, MD^*, where MB^* is the class of Lindelöfian meromorphic functions, and MD^* the class of meromorphic functions with finite spherical Dirichlet integral (see Sario and Nakai [4] for a comprehensive account of these classes).

COROLLARY 2 (CONSTANTINESCU AND CORNEA [1], KURAMOCHI [3]). *$U_{HB} \subset \mathcal{O}_{MB^*}$.*

PROOF. If $R \in U_{HB}$, it is known that the Wiener harmonic boundary contains a point p of positive harmonic measure (cf. e.g. [4]). Since the Wiener compactification R^N is resolutive, and every $f \in MB^*(R)$ has a continuous extension to R^N, the meromorphic function $f - f(p)$ vanishes at p. It follows from the Theorem that $f \equiv$ constant, and $R \in \mathcal{O}_{MB^*}$.

COROLLARY 3 (CF. SARIO-NAKAI [4]). *$U_{HD^{\sim}} \subset \mathcal{O}_{MD^*}$.*

PROOF. If $R \in U_{HD^{\sim}}$, then the Royden harmonic boundary contains a point p of positive harmonic measure. Since the Royden compactification R^M

is resolutive and every $f \in MD^*(R)$ has a continuous extension to R^M, the meromorphic function $f - f(p)$ vanishes continuously at p. As above, $f \equiv$ constant, and $R \in \mathcal{O}_{MD^*}$.

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