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# IDEMPOTENT MAXIMAL IDEALS AND INDEPENDENT SETS

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ABSTRACT. Let $E$ be a compact independent subset of a nondiscrete LCA group $G$. Let $GpE$ be the subgroup of $G$ generated algebraically by $E$. If $\mu$ is a continuous, regular, Borel measure on $GpE$ with $\mu(GpE) \neq 0$, then there exists a maximal ideal $\chi$ of the algebra $M(G)$ of regular Borel measures on $G$ such that the restriction of $\chi$ to $L^1(\mu) = \{\nu \in M(G): \nu \ll \mu\}$ is a nontrivial idempotent in $L^\infty(\mu)$. This result is used to give a new proof that $GpE$ has zero Haar measure.

**Introduction.** A subset $E \subseteq G$ of a LCA group $G$ is *independent* if $n \geq 1$, $x_1 \in G, \ldots, x_n \in G$, $m_1 \in \mathbf{Z}, \ldots, m_n \in \mathbf{Z}$ and

$$\sum_1^n m_j x_j = 0 \quad \text{imply} \quad m_1 x_1 = \cdots = m_n x_n = 0.$$

Every maximal ideal (multiplicative linear functional) $\chi$ of $M(G)$ induces an element $\chi_\mu$ of $L^\infty(\mu)$, $\mu \in M(G)$, through restriction of $\chi$ to $L^1(\mu) = \{\nu: \nu \ll \mu\}$. ($\nu \ll \mu$ means $\nu$ is absolutely continuous with respect to $\mu$.) An idempotent $f \in L^\infty(\mu)$ is a *nontrivial idempotent* if $f^{-1}(0)$ and $f^{-1}(1)$ both have nonzero $\mu$-measure. For more about maximal ideals, see **[GRS]**, **[Sr]**, **[T]**.

We prove:

THEOREM 1. *Let $E$ be a compact independent set of a LCA group $G$. Suppose $\mu$ is a continuous measure on $G$ with $\mu(GpE) \neq 0$. Then there exists a maximal ideal $\chi$ of $M(G)$ such that $\chi_\mu$ is a nontrivial idempotent in $L^\infty(\mu)$.*

COROLLARY 1. *If $H$ is a $\sigma$-compact, nondiscrete, subgroup of $G$, and $E \subseteq G$ is compact, and independent, then $(x + GpE) \cap H$ has zero $H$-Haar measure, for all $x \in G$.*

COROLLARY 2. *Let $G$ be compact, and $\mu$ a Riesz product on $G$ with* $\limsup |\hat{\mu}(\gamma)| < 1$. *If $E \subseteq G$ is compact and independent then $\mu(x + GpE) = 0$ for all $x \in G$.*

COROLLARY 3. *Let $\mu$ be any continuous measure on $G$ such that $|\chi_\mu|$ is constant a.e. $d\mu$ for all maximal ideals $\chi$ of $M(G)$. Then* $\mu(GpE) = 0$ *whenever $E$ is an independent set.*

REMARKS. (i) Corollary 1 has a long history. It was first stated in a weaker form by Rudin in [**Ru**, 5.3.6], but the proof given there is obscure. A clearer

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proof appeared in our previous note [**G**]. Later, in his 1972 Northwestern University Thesis, Rago proved Corollary 1 as a corollary of another result, most simply stated this way: if $\mu_1, \ldots, \mu_{n+1}$ are continuous measures on $G$ and $E$ is independent, then $\mu_1 * \cdots * \mu_{n+1}(n(E \cup -E)) = 0$. Salinger and Varopoulos [**SV**, Theorem 1] had proved this for $n = 1$ and metrizable $G$. Rago's result appears in his [**Ra**].

(ii) This paper grew out of an attempt to learn more about sets on which Riesz products may be concentrated. Corollary 2 is the best we have obtained so far. It seems possible that, for a Riesz product on the circle $\mu$, and a proper Borel subgroup $H$ of the circle, $\mu(H) = 0$. Corollary 2 is a trivial consequence of Theorem 1 and the deep work of Gavin Brown.

(iii) The conclusion of Corollary 3 holds for "most" infinite convolutions $\mu$ of discrete probability measures: $|\chi_\mu|$ is constant a.e. $d\mu$.

(iv) Finally, the idea of the proof of Theorem 1 is this. If $\mu(GpE) \neq 0$, then $\mu(n(E \cup -E)) \neq 0$ for some $n \geq 1$. The proof of Theorem 1 is an induction on $n$. The proof is simple when $n = 1$: we may find disjoint compact subsets $E_1$, $E_2$ of $E \cup -E$ with $\mu(E_1) \neq 0 \neq \mu(E_2)$. We let $\mathfrak{R}$ be the Raĭkov system [**GRS**] generated by $E_1$ and observe that if $\pi$ is the projection of $M(G)$ onto the $L$-algebra of measures concentrated on sets in $\mathfrak{R}$, then $\chi(\nu) = \int d(\pi\nu)$ has the required properties. The proof of Theorem 1 (for $n > 1$) is a more complicated version of this observation.

(v) The original version of this paper contained a larger and more cumbersome proof of Theorem 1. We express our thanks to the referee, who gave a simplified version, and for his (her) other helpful comments.

1. **Proof of Theorem 1.** We have several steps, in all of which we retain the hypotheses of Theorem 1. First observe that we may assume $\mu \geq 0$.

(A) Suppose $\chi_\nu$ is idempotent for all $\nu \in M(G)$. Then $\chi_\mu$ is a nontrivial idempotent iff $\chi_{\delta_x * \mu}$ is a nontrivial idempotent, for all $x \in G$. [Indeed, $\chi(\delta_x) = 1$ so $\chi_{\delta_x * \mu}(x + y) = \chi_{\delta_x}(x)\chi_\mu(y)$ a.e. $d\mu$.] The assertion now follows.

(B) Since the integers are well ordered and $\mu$ is continuous, there exists a minimal integer $m > 0$ such that (setting $Q = E \cup -E$):

$$\exists x \in G \quad \text{with } \mu(x + mQ) \neq 0 \tag{1}$$

and

$$\forall y \in G \text{ and } 0 \leq j < m, \quad \mu(y + jQ) = 0. \tag{2}$$

By using (A) we see we may assume that both

$$\mu(mQ) \neq 0 \tag{3}$$

and (2) hold. We leave $m$ fixed. We may further assume (by replacing $\mu$ with a measure absolutely continuous with respect to $\mu$) that $\mu$ has support $mQ$.

(C) Let $x_1, \ldots, x_m \in E$, with $\pm x_1 \pm \cdots \pm x_m = x$ in the support of $\mu$, for an appropriate choice of signs. Let $X = \{x_1, \ldots, x_n\}$. Then

$$mQ = \bigcup_{j=0}^{m} [j((E \backslash X \cup -(E \backslash X)) + (m - j)(X \cup -X)]. \tag{4}$$

From (2) and (4), we see that

$$(5) \qquad \mu(mQ) = \mu(m((E\backslash\{x_1, \ldots, x_m\}) \cup -(E\backslash\{x_1, \ldots, x_m\}))).$$

The regularity of $\mu$ (and continuity of addition in $G$) imply

$$(6) \qquad \mu(mQ) = \sup \mu(m((E\backslash W) \cup -(E\backslash W)))$$

where the supremum in (6) is taken over (small) neighborhoods $W$ of $\{x_1, \ldots, x_m\}$. But, if $W$ is any open neighborhood of $\{x_1, \ldots, x_m\}$, then $m[(E\backslash W) \cup -(E\backslash W)]$ misses an entire (relative) neighborhood of $\pm x_1 \pm \cdots \pm x_m = x$ in the support of $\mu$. Therefore

$$(7) \qquad \mu(mQ) > \mu(m((E\backslash W) \cup -(E\backslash W))).$$

Of course, for a sufficiently small neighborhood $W$ of $\{x_1, \ldots, x_m\}$, (6) (combined with (7)) yields

$$(8) \qquad \mu(mQ) > \mu(m((E\backslash W) \cup -(E\backslash W))) > \tfrac{1}{2}\mu(mQ) > 0.$$

(D) We do some computations with $mQ$. Assume $W$ (a neighborhood of $\{x_1, \ldots, x_m\}$) has been chosen so (8) holds.

For a set $F \subseteq G$, and $j \geq 0$, set $(j) * F = j(F \cup -F)$, and $(0) * F = \{0\}$. Then

$$(9) \qquad mQ = \bigcup_{j=0}^{m} (j) * (E\backslash W) + (m-j) * E \cap W$$

and (8) and (9) together imply that for some $0 \leq j_1 < m$,

$$(10) \qquad \mu((j_1) * (E\backslash W) + (m - j_1) * (E \cap W)) > 0.$$

(E) Let $\mathfrak{R}$ be the Raĭkov system in $G$ which is generated by $F \cup -F$ where $F = E\backslash W$. Let $\pi$ be the map from $M(G)$ to $A_{\mathfrak{R}}$, the algebra of measures $\omega$ on $G$ such that

$$(11) \qquad \begin{aligned} \|\omega\| = \sup\{|\omega|(y_1 + GpF \cup \cdots \cup y_j + GpF)&: \\ y_1, \ldots, y_j \in G, 1 \leq j < \infty\}&. \end{aligned}$$

(Facts about $A_{\mathfrak{R}}$ and $\pi$ can be found in [**GRS**].)

Then the definition of $\pi$, (8), and (11) together imply

$$\|\pi\mu\| \geq \mu((m) * F) = \mu(m((E\backslash W) - (E\backslash W))) > 0.$$

Let $\chi$ be the maximal ideal of $M(G)$ defined by

$$\chi(\omega) = (\pi\omega)\hat{}\,(0); \qquad \omega \in M(G).$$

Straightforward computations show that

$$(12) \qquad \begin{cases} \chi_{\pi\omega} \equiv 1 & \text{a.e. } d\pi\omega, \\ \chi_{(\omega-\pi\omega)} \equiv 0 & \text{a.e. } d(\omega - \pi\omega), \end{cases}$$

so $\chi_\omega$ is an idempotent for all $\omega \in M(G)$. In particular $\chi(\mu) = \|\pi\mu\| > 0$, so formula (12), applied to $\mu$ shows (using (11)) that $\chi_\mu = 1$ on a set of nonzero $\mu$-measure. We must show that $\chi_\mu = 0$ on another set of nonzero $\mu$-measure. This will complete the proof that $\chi_\mu$ is a nontrivial idempotent.

(F) Set $K = (j_1) * F + (r) * L$, where $r = m - j_1$, and $L = E \cap W$. We wish to prove

$$\mu(K \cap (y + GpF)) = 0 \quad \text{for all } y \in G. \tag{13}$$

Then (11) and (13) imply $\pi\mu(K) = 0$ while (10) implies $\mu(K) > 0$. This shows that $\chi_\mu(x) \equiv 0$ a.e. $d\mu$ for $x \in K$.

So suppose $K \cap (y + GpF) \neq \varnothing$, for some $y \in G$. Then (from the definition of $K$), there exist $x \in (r) * L$, and $a \in GpF$ such that $y = x + a$. If $z \in K \cap (y + GpF)$, then for some $b \in GpF$, $c \in (j_1) * F$, $d \in (r) * L$,

$$z = c + d = y + b = (a + b) + x.$$

Since $E$ is independent, $(GpF) \cap (GpL) = \varnothing$, so $c = a + b$ and $d = x$. Therefore

$$z = x + c \in x + (j_1) * F.$$

Therefore

$$K \cap (y + GpF) \subseteq x + (j_1) * F,$$

which has $\mu$-measure zero, by (2).

## 2. **Proof of corollaries.**

Proof of Corollary 1. If $\lambda$ is $H$-Haar measure restricted to any set of finite $H$-Haar ($\lambda$) measure, and $\chi$ is any maximal ideal of $M(G)$, then $\chi_\lambda$ may not be a nontrivial idempotent, since $\chi_\lambda$, on $L^1(\lambda) = \{\nu \in M(G)\colon \nu$ is absolutely continuous with respect to $\lambda\}$, is either zero, or agrees $\lambda$-almost everywhere with a (unimodular) character on $H$. This proves the corollary when $x = 0$. Now apply paragraph (A) of the proof of Theorem 1.

Proof of Corollary 2. Brown **[B]** shows each $\chi$ has $|\chi_\mu|$ constant a.e. $d\mu$ if $\mu$ is a Riesz product with $\limsup |\hat{\mu}(\gamma)| < 1$.

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