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APPROXIMATION TO FIXED POINTS OF GENERALIZED NONEXPANSIVE MAPPINGS

CHI SONG WONG[1]

ABSTRACT. Let K be a convex subset of a uniformly convex Banach space. It is proved that if K is compact, then the fixed points of a continuous generalized nonexpansive self-mapping T on K can be approximated by the iterates of T_t with $t \in (0, 1)$, $T_t(x) = (1 - t)x + tT(x)$, $x \in K$; T_t is asymptotically regular if T has a fixed point.

Let (X, d) be a (nonempty) metric space. A function α of $X \times X$ into $[0, \infty)$ is *symmetric* if $\alpha(x,y) = \alpha(y,x)$ for all x,y in X. Let T be a self-mapping on X. T is *generalized nonexpansive* if there exist symmetric functions α_i, $i = 1, 2, \ldots, 5$, of $X \times X$ into $[0, \infty)$ such that

$$\sup\left\{\sum_{i=1}^{5} \alpha_i(x,y) : x,y \in X\right\} \leqslant 1 \tag{1}$$

and for all x, y in X,

$$\begin{aligned} d(T(x), T(y)) \leqslant{}& a_1 d(x,y) + a_2 d(x, T(y)) + a_3 d(y, T(x)) \\ &+ a_4 d(x, T(x)) + a_5 d(y, T(y)), \end{aligned} \tag{2}$$

where $a_i = \alpha_i(x,y)$. It is clear that T is generalized nonexpansive if it is nonexpansive $(d(T(x), T(y)) \leqslant d(x,y), x,y \in X$. R. Kannan first considered those T which satisfy (2) with $a_1 = a_2 = a_3 = 0$ and $a_4 = a_5 \leqslant \frac{1}{2}$ [5]–[9]. S. Reich considered those T which satisfy (2) with $a_2 = a_3 = 0$ and with constants a_1, a_4, a_5 [11]–[13]. Recently, G. Hardy and T. Rogers considered those T which satisfy (2) with constants a_i's [4]. In [3], K. Goebel, W. A. Kirk and Tawfik N. Shimi proved that T has a fixed point if X is a weakly compact convex subset of a uniformly convex Banach space and if T satisfies (2) with constant coefficients. Other related work can be found in [15]–[19]. In this paper, we shall investigate the theory of approximations to fixed points of generalized nonexpansive mappings.

1. **Asymptotic regular mappings.** Let T be a self-mapping on a metric space

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(X,d). T is asymptotically regular [1] if for any x in X, $\{d(T^{n+1}(x), T^n(x))\}$ converges to 0. For a Banach space B, we shall use d to denote the metric for B induced by the norm $\| \ \|$ of B.

THEOREM 1. *Let X be a convex subset of a uniformly convex Banach space B. Let T be a generalized nonexpansive self-mapping on X. Suppose that the fixed point set $F = \{x \in X\colon T(x) = x\}$ is nonempty. Then for each t in $(0, 1)$, the mapping T_t defined by*

$$T_t(x) = (1 - t)x + tT(x), \qquad x \in X, \tag{3}$$

is asymptotically regular.

PROOF. Let $x_0 \in X$, $x_{n+1} = T_t(x_n)$, $n = 0, 1, 2, \ldots$. Since

$$T_t(x) - x = t(x - T(x)), \qquad x \in X, \tag{4}$$

it suffices to prove that $\{d(x_n, T(x_n))\}$ converges to 0. Since T is a generalized nonexpansive mapping, there exist symmetric functions α_i, $i = 1, 2, \ldots, 5$, of $X \times X$ into $[0, \infty)$ such that $\sum_{i=1}^{\infty} \alpha_i \leqslant 1$ and for all x, y in X,

$$\begin{aligned} d(T(x), T(y)) \leqslant a_1 d(x,y) + a_2 d(x, T(y)) + a_3 d(y, T(x)) \\ + a_4 d(x, T(x)) + a_5 d(y, T(y)), \end{aligned} \tag{5}$$

where $a_i = \alpha_i(x,y)$. Each α_i is symmetric. So we may, by calculating $(d(T(x), T(y)) + d(T(y), T(x)))/2$, assume that $\alpha_2 = \alpha_3$, $\alpha_4 = \alpha_5$. Thus $\alpha_3 + \alpha_4 \leqslant \frac{1}{2}$. Now let $x \in X$, $y \in F$. From (5) (with $a_i = \alpha_i(x,y)$),

$$\begin{aligned} d(T(x),y) &= d(T(x), T(y)) \\ &\leqslant (a_1 + a_2)d(x,y) + a_3 d(y, T(x)) + a_4 d(x, T(x)). \end{aligned}$$

Since $d(x, T(x)) \leqslant d(x,y) + d(y, T(x))$,

$$(1 - a_3 - a_4)d(T(x),y) \leqslant (a_1 + a_2 + a_4)d(x,y).$$

Since $1 - a_3 - a_4 > 0$ and $1 - a_3 - a_4 \geqslant a_1 + a_2 + a_4$,

$$d(T(x),y) \leqslant d(x,y). \tag{6}$$

Also

$$\begin{aligned} \|y - T_t(x)\| &= \|y - (1 - t)x - tT(x)\| \\ &= \|(1 - t)(y - x) + t(y - T(x))\| \end{aligned} \tag{7}$$

$$\leqslant (1 - t)\|y - x\| + t\|y - T(x)\|. \tag{8}$$

By (6) and (8),

$$\|y - T_t(x)\| \leqslant \|y - x\|. \tag{9}$$

so $\{\|y - x_n\|\}$ is bounded by $M = \|y - x_0\|$. If $y = x_n$ for some n, then from (9), $\{x_n\}$ converges to y and the proof is complete. So we may assume that

$y \neq x_n$ for all $n = 0, 1, 2, \ldots$. We shall now assume that $t \leqslant \frac{1}{2}$. Note first that

$$\begin{aligned} \|y - x_{n+1}\| &= \|t(y - x_n + y - T(x_n)) + (1 - 2t)(y - x_n)\| \\ &\leqslant t\|y - x_n + y - T(x_n)\| + (1 - 2t)\|y - x_n\| \\ &\leqslant 2t\|y - x_n\| \, \|(u + v)/2\| + (1 - 2t)\|y - x_n\|, \end{aligned} \tag{10}$$

where

$$u = \frac{y - x_n}{\|y - x_n\|}, \qquad v = \frac{y - T(x_n)}{\|y - x_n\|}. \tag{11}$$

Since B is uniformly convex,

$$\delta(\epsilon) = \inf\{1 - \|x + y\|/2 \colon \|x\| \leqslant 1, \|y\| \leqslant 1, \|x - y\| \geqslant \epsilon\}$$

is positive for ϵ in (0,2]. Also $\delta(0) = 0$. From (11),

$$\|(u + v)/2\| \leqslant 1 - \delta(\|x_n - T(x_n)\|/\|y - x_n\|).$$

Since δ is monotonically nondecreasing on [0,2],

$$\|(u + v)/2\| \leqslant 1 - \delta(\|x_n - T(x_n)\|/M). \tag{12}$$

From (10) and (12),

$$\|y - x_{n+1}\| \leqslant (1 - 2t\delta(\|x_n - T(x_n)\|/M))\|y - x_n\|. \tag{13}$$

By (13) and induction,

$$\|y - x_{n+1}\| \leqslant \prod_{j=0}^{n} (1 - 2t\delta(\|x_j - T(x_j)\|/M))M. \tag{14}$$

Suppose to the contrary that $\{\|x_n - T(x_n)\|\}$ does not converge to 0. Then there exists a subsequence $\{x_{k(n)}\}$ of $\{x_n\}$ such that $\{\|x_{k(n)} - T(x_{k(n)})\|\}$ converges to some $r \in (0, \infty)$. Since δ is monotonically nondecreasing and $1 - 2t\delta(\|x_j - T(x_j)\|/M) \in [0, 1]$ for each j, we have from (14) that for large n's,

$$\|y - x_{k(n+1)}\| \leqslant (1 - 2t\delta(r/2M))^n M.$$

So $\{x_{k(n)}\}$ converges to y. By (6), $\{T(x_{k(n)})\}$ also converges to y. Thus $\{\|x_{k(n)} - T(x_{k(n)})\|\}$ converges to 0, a contradiction to the choice of r. If $t > \frac{1}{2}$, then $1 - t \leqslant \frac{1}{2}$ and we can repeat the same argument by writing (10) as

$$\begin{aligned} \|y - x_{n+1}\| &= \|(1 - T)(y - x_n + y - T(x_n)) + (2t - 1)(y - T(x_n))\| \\ &\leqslant (1 - t)\|y - x_n + y - T(x_n)\| + (2t - 1)\|y - T(x_n)\| \\ &\leqslant (1 - t)\|y - x_n\| \, \|(u + v)/2\| + (2t - 1)\|y - x_n\|; \end{aligned}$$

a contradiction will then be obtained as before if we interchange the roles of

t and $1 - t$. Hence T_t is asymptotically regular.

The above result was proved in [14] for nonexpansive mappings.

2. **Successive approximations.**

THEOREM 2. *Let X be a compact convex subset of a strictly convex Banach space B. Let T be a generalized nonexpansive self-mapping on X. Then*:

(*a*) *The fixed point set $F = \{x \in X: T(x) = x\}$ of T is convex.*

(b) *If T is continuous, then F is nonempty and compact.*

(c) *If T is continuous, then for any x_0 in X, t in* $(0,1)$, $\{T_t^n(x_0)\}$ *converges to a fixed point of T, where $T_t(x) = (1 - t)x + tT(x)$, $x \in X$.*

PROOF. (a) Let $x, y \in F$, $t \in (0,1)$, and, and $z = (1 - t)x + ty$. Then by (6), $d(T(z),x) \leqslant d(z,x) = td(x,y)$. Similarly, $d(T(z),y) \leqslant (1 - t)d(x,y)$. By the strict convexity of B, $T(z) = z$. Hence F is convex.

(b) By continuity and the Schauder-Tychonoff theorem, F is compact and nonempty.

(c) Let $n \geqslant 0$, $x_n = T_t^n(x_0)$. Since X is compact, $\{x_n\}$ has a subsequence $\{x_{k(n)}\}$ which converges to some point x in X. We shall prove that x is a fixed point of T. From (9), $\{d(x_n,y)\}$ is monotonically nonincreasing. So by continuity of $\| \ \|$ and T_t,

$$\begin{aligned} \|x - y\| &= \lim_{n\to\infty} \|x_{k(n+1)} - y\| \leqslant \lim_{n\to\infty} \|x_{k(n)+1} - y\| \\ &= \lim_{n\to\infty} \|T_t(x_{k(n)}) - y\| = \|T_t(x) - y\|. \end{aligned} \tag{15}$$

By (9) and (15),

$$\|T_t(x) - y\| = \|x - y\|. \tag{16}$$

Note that

$$\begin{aligned} \|T_t(x) - y\| &= \|(1 - t)x + tT(x) - y\| \\ &= \|(1 - t)(x - y) + t(T(x) - y)\| \\ &\leqslant (1 - t)\|x - y\| + t\|T(x) - y\| \\ &\leqslant (1 - t)\|x - y\| + t\|x - y\| = \|x - y\|. \end{aligned} \tag{17}$$

Combining (16) and (17), we conclude that all inequalities in (17) are equalities. So

$$\|(1 - t)(x - y) + t(T(x) - y)\| = (1 - t)\|x - y\| + t\|T(x) - y\| \tag{18}$$

and

$$\|x - y\| = \|T(x) - y\|. \tag{19}$$

By (19) and the strict convexity of B, either $x = y$ or $T(x) - y = s(x - y)$ for some $s > 0$. From (19), $s = 1$. So x is a fixed point of T. By (18), $\{d(x_n, x)\}$ is monotonically nondecreasing. Hence $\{x_n\}$ converges to x.

The above result was proved in [2] for nonexpansive mappings.

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DEPARTMENT OF MATHEMATICS, UNIVERSITY OF WINDSOR, WINDSOR, ONTARIO, CANADA