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THE FUNDAMENTAL THEOREM OF ALGEBRA ON RATIONAL H-SPACES

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ABSTRACT. A form of the Fundamental Theorem of Algebra is proven for Γ-H structures.

0. **Introduction and summary of results.** It is a curiosity that while the Fundamental Theorem of Algebra is algebraic in content and statement, its proof is topological. We show here that the structure of a rational H-space, or Γ structure is sufficient to prove a form of the theorem.

In §1 we construct a rational H-space structure on the Stiefel manifold $V_{5,2}$ which we believe has not been explicitly exhibited before. In §2 we prove that any homotopy associative rational H-space is actually an H-space.

1. **Rational H-spaces or Γ structures.** A rational H-space is a triple (X,m,e) consisting of a connected space X, a product map $m : X \times X \to X$ which preserves the basepoint e, and which satisfies the following condition. Let $\varphi_i : X \to X \times X$ be inclusion into the ith factor, $i = 1, 2$. Then $m_l = m\varphi_1$ and $m_r = m\varphi_2$ must induce automorphisms of $H^*(X; \mathbf{Q})$.

If $m_l \simeq m_r \simeq \mathrm{id}$, then e is a homotopy identity and (X,m,e) is actually an H-space.

For notational simplicity we will follow Hopf [4] and refer to rational H-spaces as Γ structures.

EXAMPLES. (1) (Hopf [4]). For an odd dimensional sphere define $m : S^n \times S^n \to S^n$ by $m(p, q) = q$ reflected through the orthogonal complement of p. Then $m_r^*(z) = -z$ and $m_l^*(z) = 2z$ where z generates $H^*(S^n)$.

(2) Consider the Stiefel manifold $V_{n,2}$ of 2 frames in $\mathbf{R}^n$ for odd n. $V_{n,2}$ can be fibered as an $n-2$ sphere bundle over S^{n-1} [5]. Since $\Pi_{2n-3}(S^{n-1})$ has a cyclic infinite subgroup [3, p. 74], and $\Pi_k(S^{n-2})$ is finite whenever $k \neq n-2$ [7, p. 515], it is clear from the long exact sequence of the fibration that $\Pi_{2n-3}(V_{n,2})$ has a cyclic infinite subgroup, with generator, say g.

Let μ be a generator of $H^{2n-3}(S^{2n-3}, \mathbf{Z})$. Then $g_*(\mu) = h([g])$ where $h : \Pi_{2n-3}(V_{n,2}) \to H_{2n-3}(V_{n,2}; \mathbf{Z})$ is the Hurewicz homomorphism. Since $H^*(V_{n,2}; \mathbf{Q}) = \Lambda(z_{2n-3}, 1)$, an exterior algebra on odd dimensional generators [6], we can apply a result of Arkowitz and Curjel [1] to conclude that $g^*(\mu) \neq 0$.

The cell structure of $V_{n,2}$ is known to be $S^{n-2} \cup e^{n-1} \cup e^{2n-3}$. Let $\rho : V_{n,2}$

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$\to S^{2n-3}$ be the collapsing map. Then the composition

$$V_{n,2} \times V_{n,2} \xrightarrow{\rho\times\rho} S^{2n-3} \times S^{2n-3} \xrightarrow{m} S^{2n-3} \xrightarrow{g} V_{n,2}$$

is a Γ structure on $V_{n,2}$ where m is Hopf's Γ structure.

It is well known that if a CW complex X admits a Γ structure, then $H^*(X,\mathbf{Q}) = \Lambda_{\nu\in N}\{z_\nu\} \otimes P_{\mu\in M}\{x_\mu\}$ where $\Lambda\{z_\nu\}$ is an exterior algebra on odd dimensional generators and $P\{x_\mu\}$ is a polynomial algebra on even dimensional generators. Furthermore, for $y \in H^*(X,\mathbf{Q})$,

$$m^*(y) = m_l^*(y) \otimes 1 + \sum y' \otimes y'' + 1 \otimes m_r^*(y)$$

with $\deg y' > 0$.

2. **Words on Γ structures.** The product m induces a binary operation, called convolution, on the set of maps from X to itself. $f \circ g = m \circ fxg \circ \Delta$ where $\Delta(x) = (x,x)$, the usual diagonal. A word on X is the convolution of a finite number of identity and constant maps. The identity map on X will be denoted by ι, the constant map at p by ω_p or simply ω. Other identity maps will be denoted by id.

THEOREM 1.1. *If (X,m,e) is a homotopy associative Γ structure, then X admits an H structure.*

PROOF. It suffices to show that $m_r^* = m_l^* = \text{id}$. Then, by Whitehead's theorem, m_r and m_l are homotopy equivalences and $m \circ (m_l^{-1} \times m_r^{-1})$ is an H structure.

If m is homotopy associative, then, in particular, $(\omega \cdot \omega)^* \cdot \iota = \omega^* \cdot (\omega \cdot \iota)^*$. Direct computation shows that the left-hand side is m_r^* and the right side is $m_r^* \circ m_r^*$. A similar computation holds for m_l^*.

We adopt the technical convention that all words will be spelled correctly. That is, convolution is always from the left, i.e.,

$$g_1 \cdot g_2 \cdot \ldots \cdot g_r = g_1 \cdot (g_2 \cdot (\cdots (g_{r-1} \cdot g_r)) \cdots)$$

and no two consecutive maps are constants. (This is always possible since $\omega_p \cdot \omega_q = \omega_{m(p,q)}$.)

Let $\{y_j\}$ denote the canonical basis for $H^*(X;\mathbf{Q})$ as a vector space over $\mathbf{Q}$ corresponding to $\{z_\nu\}_{\nu\in N}$ and $\{x_\mu\}_{\mu\in M}$. A typical element is of the form $y_j z_{\nu_1} z_{\nu_2} \cdots z_{\nu_k} x_{\mu_1} \cdots x_{\mu_r}$. For each basis element y_j we define an integer

$$\lambda(y_j) = \max\{\text{degree } z_{\nu_i}, \text{degree } x_{\mu_j}, i = 1,\ldots,k; j = 1,\ldots,r\}.$$

Let $G(X)$ be the vector space over $\mathbf{Q}$ spanned by $\{z_\nu\}_{\nu\in N} \cup \{x_\mu\}_{\mu\in M}$. Define $\alpha(z_\nu)$ and $\beta(z_\nu)$ to be the summand of $m_l^*(z_\nu)$ and $m_r^*(z_\nu)$ which lives in $G(X)$, i.e., if $m_l^*(z_\nu) = \sum q_i z_{\nu_i} + \sum a_j y_j$, $a_i a_j \in \mathbf{Q}$ and $\lambda(y_j) <$ degree z_ν, then $\alpha(z_\nu) = \sum a_i z_{\nu_i}$.

Associated to every word f on X we define an automorphism of $G(X)$ called the exponent of f, $\epsilon(f)$.

$$\begin{aligned} \epsilon(f) &= \text{id} && \text{if } f = \iota, \\ &= 0 && \text{if } f = \omega, \\ &= \alpha + \epsilon(g) \circ \beta && \text{if } f = \iota \cdot g, \\ &= \epsilon(g) \circ \beta && \text{if } f = \omega \cdot g. \end{aligned}$$

If (X,m,e) is an H-space, then $\alpha = \beta = \mathrm{id}$ and $\epsilon(f)$ coincides with the definition of exponent given by R. F. Brown [2].

LEMMA 2.1. *If $f : X \to X$ is a word, then $f^*(z_\nu) = \epsilon(f)z_\nu + \sum a_j y_j$ where $\lambda(y_j) <$ degree z_ν.*

PROOF. The result is clear if $f = \iota$ or ω. We proceed by induction on the number of maps convoluted to form f. If $f = \iota \cdot g$, then $f^*(z_\nu) = \alpha(z_\nu) + \sum a_j y_j + g^*\beta(z_\nu)$, $\beta(z_\nu) = \sum b_i z_{\nu_i}$, $b_{i \in \mathbf{Q}}$, degree $z_{\nu_i} =$ degree z_ν. Hence

$$g^*\beta(z_\nu) = \sum b_i g^*(z_{\nu_i}) = \sum b_i(\epsilon(g) z_{\nu_i} + \sum a_j y_j)$$
$$= \epsilon(g) \circ \beta(z_\nu) + \sum a_j y_j$$

with $\lambda(y_j) <$ degree z_ν. The case $f = \omega \cdot g$ is a similar computation.□

3. Γ-H **structures.** Following [2] we define $C^0(X)$, the open cone on X, as $X \times [0, \infty)$ with $X \times \{0\}$ collapsed to a point. Square brackets will be used to denote equivalence classes and $[x, 0]$ will be written as 0. Any Γ structure (X,m,e) can be extended to a Γ structure $(C^0(X), c(m), [e, 1])$ by defining $c(m)([x, t], [y, s]) = [m(x, y), ts]$.

A Γ-H structure (X,m,e,η) is a Γ structure (X,m,e) together with an H structure on $C^0(X)$ for which 0 is the basepoint, i.e., $\eta : C^0(X) \times C^0(X) \to C^0(X)$ and $\eta([x,t], 0) = \eta(0, [x,t]) = [x,t]$. We will denote the binary operation η induces on words of the Γ structure $(C^0(X), c(m), [e, 1])$ by ♀. A polynomial on $C^0(X)$ is a map from $C^0(X)$ to itself of the form g_1 ♀ g_2 ♀ $\cdots$ ♀ g_k, where g_i is a word on $(C^0(X), c(m), [e, 1])$. $[x, t]$ is a root of a polynomial if $f[x, t] = 0$.

EXAMPLE 3.1. Let $X = S^1$ considered as complex numbers of unit norm, m is standard complex multiplication, η is usual addition. $\omega_{a_n} \cdot \iota^n$ ♀ $\omega_{a_{n-1}} \cdot \iota^{n-1}$ ♀ $\cdots$ ♀ ω_{a_0} is the familiar polynomial $a_n x^n + a_{n-1}x^{n-1} + \cdots + a_0$.

Given a polynomial $f = g_1$ ♀ g_2 ♀ $\cdots$ ♀ g_k, define a polynomial

$$f_{is} = \psi(s, g_1) \text{ ♀ } \cdots \text{ ♀ } \psi(s, g_{i-1}) \text{ ♀ } g_i \text{ ♀ } \psi(s, g_{i+1}) \text{ ♀ } \cdots \text{ ♀ } \psi(s, g_k),$$

i.e. apply $\psi(s,\ )$ to each word except g_i where

$$\psi(s, g)[x, t] = \theta(s, g[x, t]) \quad \text{and} \quad \theta(s, [x, t]) = [x, st].$$

The polynomial f is admissible if $f_{i,s}$ is proper for some i uniformly in s, i.e. $f_{i,s}$ extends to the suspension SX viewed as the one point compactification of $C^0(X)$. Not all nonconstant words are admissible.

EXAMPLE 3.2. Consider the quaternions as $C^0(S^3)$. Then the polynomial $\omega_{[i,1]}$ ♀ $\iota \cdot \omega_{[i,1]}$ ♀ $\omega_{[i,1]} \cdot \iota$ is not admissible and does not have a root.

LEMMA 3.3. *If $f = g_1$ ♀ $\cdots$ ♀ g_k is an admissible polynomial with $f_{i,s}$ proper, then $\tilde{f} \simeq \tilde{g}_i$, where tilde denotes the extension to SX.*

PROOF. Define $H : SX \times I \to SX$ by $H([x, t], s) = \tilde{f}_{is}[x, t]$.

A word g on (X, m, e) is compatible with m if $\epsilon(g) \neq 0$. A polynomial $f = g_1$ ♀ $\cdots$ ♀ g_k is compatible if f is admissible and g_i is compatible with $c(m)$. Define a word $\bar{g}_i$ on (X, m, e) by $\bar{g}_i(x) = g_i[x, 1]$. Then by 3.3, $\tilde{f} \simeq \tilde{g}_i \simeq \sum \bar{g}_i$.

EXAMPLE 3.4. Let $m : S^1 \times S^1 \to S^1$ be given by $m(e^{i\theta}, e^{i\varphi}) = e^{i(\theta-\varphi)}$. The polynomial $\iota^2 \, ♀ \, \omega_{[e^{i\pi/2},1]}$ is admissible but not compatible. In this case $g_i = \iota^2$ and $\epsilon(g) = 1 - 1 = 0$. It is not hard to show that f is rootless.

4. **Fundamental Theorem of Algebra.**

THEOREM. (X, m, e, η) *is a* Γ-H *structure with* $H^*(X; \mathbf{Q}) = \Lambda\{z_\nu\} \otimes P\{x_\mu\}$. *Every polynomial on* $C^0(X)$ *which is compatible with m has a root.*

PROOF. Suppose $f = g_1 \, ♀ \, g_2 \, ♀ \cdots ♀ \, g_k$. By compatibility $\tilde{f} \simeq \bar{g}_i$ and $\epsilon(\bar{g}_i) \neq 0$. $\bar{g}_i^*(z_\nu) = \epsilon(\bar{g}_i)z_\nu + \sum a_j y_j \neq 0$ by Lemma 2.1. Then $\tilde{f}^*\sigma(z_\nu) = S\bar{g}_i^*(\sigma z_\nu) = \sigma\bar{g}_i^*(z_\nu) \neq 0$ where $\sigma : H^*(X) \to H^*(SX)$ is the suspension homomorphism.

On the other hand if $\tilde{f}$ is not onto then its image will be contained in a contractible space and $\tilde{f}^* = 0$.

Note this generalizes Theorem 2 of [2].

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