PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 54, January 1976

A GEOMETRIC INTERPRETATION OF A CLASSICAL GROUP COHOMOLOGY OBSTRUCTION

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ABSTRACT. For a non-Abelian group G, we show that the obstruction to the existence of an extension of G by Π that induces $\phi\colon \Pi \to \operatorname{Out} G$ is also the k-invariant of the classifying space for $K(G, 1)$-bundles.

1. In the classical theory of group extensions, there arises a group cohomology obstruction. This paper studies some topological implications of that obstruction, giving it a geometric interpretation and augmenting the classification of fibrations by J. Siegal [15], [16] with the case the fiber is a $K(G, 1)$, G not Abelian.

If $1 \to G \to E \to \Pi \to 1$ is an extension of the group G by the group Π, it induces either an action $\phi\colon \Pi \to \operatorname{Aut} G$, if G is Abelian, or a "semiaction" $\phi\colon \Pi \to \operatorname{Out} G = \operatorname{Aut} G/\operatorname{In} G$, otherwise. See [9, p. 124]. Given G not Abelian and a $\phi\colon \Pi \to \operatorname{Out} G$, there need not exist an extension inducing ϕ. Indeed, let C be the center of G and $\alpha\colon \operatorname{Out} G \to \operatorname{Aut} C$ be induced by restriction $\operatorname{Aut} G \to \operatorname{Aut} C$. By [4] or [9, p. 128] we have

THEOREM A. *There is an obstruction $k \in H^3_{\alpha\phi}(\Pi; C)$ to the existence of an extension of G by Π which induces ϕ. If $k = 0$, the set of all equivalence classes of all such extensions is in a* 1-1 *correspondence with $H^2_{\alpha\phi}(\Pi; C)$.*

In §4 this obstruction is given in detail, and we prove the following, which shows this obstruction has a "universal example."

THEOREM B. *Let $U \in H^3_\alpha(\operatorname{Out} G; C)$ be the obstruction to the existence of an extension of G by* $\operatorname{Out} G$ *which induces* $\mathrm{id}\colon \operatorname{Out} G \to \operatorname{Out} G$. *Then for any $\phi\colon \Pi \to \operatorname{Out} G$, $\phi^*(U)$ is the obstruction k of Theorem* A.

Recall, even though $K(G, 1)$ is not an H-space (for G not Abelian), there is a universal classifying fibration $K(G, 1) \to E \xrightarrow{p} B$. By Gottlieb [6, p. 54], $\pi_1(B) \cong \operatorname{Out} G$, $\pi_2(B) \cong C$ and $\pi_i(B) = 0$, otherwise. Thus B has a single (twisted) k-invariant in $H^3(K(\operatorname{Out} G, 1); \{G\})$ (where $\{\ \}$ denotes local coefficients). We denote by $\Phi\colon H^*_{\alpha\phi}(\Pi; C) \to H^*_{\alpha\phi}(K(\Pi, 1); \{C\})$ the natural isomorphism (see [9, IV. 11]). The main result of this paper is

THEOREM C. *Let U be given by Theorem* B. *Then $\Phi(U)$ is the (twisted) k-invariant for the universal classifying space for $K(G, 1)$ bundles.*

Theorem A has a natural geometric consequence. Given an extension $G \xrightarrow{i} E \xrightarrow{p} \Pi$, we can construct a fibration $K(G, 1) \xrightarrow{i'} K(E, 1) \xrightarrow{p'} K(\Pi, 1)$

Received by the editors November 1, 1974.
AMS (MOS) subject classifications (1970). Primary 55F15; Secondary 55G35, 18H10, 20F25.

in which $i'_* = i$ and $p'_* = p$ on the fundamental groups. In any fibration, "dragging" the fiber around loops in the base (backwards) induces an action of π_1(base) on π_n(fiber), but only modulo the action of π_1 on π_n. Thus, the fibration p' induces a geometric "semiaction" π_1 (base) $\to$ Out (π_1 (fiber)) in addition to the algebraic one.

THEOREM D. *These two "semiactions" are the same.*

This may be proven by extending either the proof of Theorem 1(a) in [7] or [14, pp. 86–88]

Using Φ we therefore easily get as a corollary of Theorem A:

THEOREM E. *Given a* $\phi\colon \Pi \to \operatorname{Out} G$, *there is an obstruction*

$$k \in H^3_{\alpha\phi}(K(\Pi, 1); \{C\})$$

to the existence of a $K(G, 1)$*-fibration over* $K(\Pi, 1)$ *which induces* ϕ. *If* $k = 0$, *the set of all equivalence classes of such fibrations is in a* 1-1 *correspondence with* $H^2_{\alpha\phi}(K(\Pi, 1); \{C\})$.

Although Theorem A gives a proof of this, it does not give any geometrical insight to this geometric fact. Theorem C leads to a very satisfactory geometric proof, which is given in §3.

The proof of Theorem C yields a complete description of p.

THEOREM F. *Let* $K(G, 1) \to E \xrightarrow{p} B$ *be the universal* $K(G, 1)$*-fibration. Then* E *is a* $K(\operatorname{Aut} G, 1)$, *its homotopy sequence reduces to the natural* $0 \to C \to G \to \operatorname{Aut} G \to \operatorname{Out} G \to 1$, *and the "semiaction" of* π_1 *(base) on* π_1 *(fiber) corresponds to the identity*: $\operatorname{Out} G \to \operatorname{Out} G$.

Theorems C and F follow from 3.1 and 3.2.

I would like to thank Professor William Massey who first posed to me the problem of geometrically understanding Theorem A.

I understand recently that some of the above with somewhat different proofs have been known to M. G. Barratt.

As a matter of convenience (see the second half of [3]), the remainder of this paper is done in the category of semisimplicial complexes. The reader is assumed to be familiar with May [10], as notation tends to be based on that text.

2. We give a concise development of $L_\pi(C, n)$ and its classifying properties. For more details see [5], [8], [11], [12], [13], [15], [16], [17].

We will denote an Eilenberg-Mac Lane complex of type (C, n) by $K(C, n)$, its "universal class" in $H^n(K(C, n); C)$ by V and the universal principal "loop-path" fibration by $K(C, n-1) \to L(C, n) \xrightarrow{P} K(C, n)$. Let $W(\Pi)$ denote the standard free acyclic semisimplicial complex corresponding to the group Π. If Π acts on C *via* $\phi\colon \Pi \to \operatorname{Aut} C$, then Π acts on both $K(C, n)$ and $L(C, n)$ naturally. Using the diagonal action, we denote $W(\Pi) \times K(C, n)/\Pi$ by $L_\pi(C, n)$ and $W(\Pi) \times L(C, n)/\Pi$ by $T_\pi(C, n)$, and obtain from P a Kan fibration $K(C, n-1) \to T_\pi(C, n) \xrightarrow{P} L_\pi(C, n)$. (The inclusion of $W(\Pi) \times *$ into $W(\Pi) \times K(C, n)$ induces a natural $K(\Pi, 1)$ contained in $L_\pi(C, n)$.)

THEOREM 2.1. *Let $n \geq 2$. (1) $\pi_i(L_\pi(C,n))$ is Π, if $i = 1$; C, if $i = n$; 0, otherwise; and the action of π_1 on π_n is ϕ.*

(2) There is a universal class $v \in H_\phi^n(L_\pi(C,n),\{C\})$, where this denotes cohomology with local coefficients twisted by ϕ. If X is a semisimplicial complex, then $f \leftrightarrow f^*(v)$ is a 1-1 correspondence between the set of based homotopy classes of maps from X into $L_\pi(C,n)$ which induce

$$\beta\colon \pi_1(X) \to \pi_1(L_\pi(C,n)) \quad \text{and} \quad H^n_{\phi\beta}(X;\{C\}).$$

(3) A map $f\colon X \to L_\pi(C,n)$ lifts if and only if $f^*(v) = 0$ if and only if f is homotopic rel *base point to a map whose image is in $K(\Pi,1)$. If $f^*(v) = 0$, the set of homotopy classes of such liftings is in 1-1 correspondence with $H^{n-1}_{\phi\beta}(X;\{C\})$.*

Parts (1) and (2) are proved by Gitler [5], and part (3) is essentially proven in Nussbaum [11], [12] and Siegel [16].

The space $L_{\mathrm{Aut}C}(C,n)$ is understood to have the natural action of π_1 on π_n.

Suppose a Kan complex has two nonzero homotopy groups, $\pi_1(X) = \Pi$, $\pi_n(X) = C$, and π_1 acts on $\pi_n(X)$ with $\phi\colon \Pi \to \mathrm{Aut}\,C$. Then X can be constructed, up to homotopy, as the pullback from the fibration p by a map $f\colon K(\Pi,1) \to L_{\mathrm{Aut}\,C}(C,n+1) = L$ such that $\phi = f_*\colon \pi_1(X) \to \pi_1(L)$.

DEFINITION. The element $f^*(v) \in H_\phi^{n+1}(K(\Pi,1);\{C\})$ is the (twisted) k-invariant of X.

As a simple illustration, we observe

PROPOSITION 2.2. *Let $n \geq 2$. The complex X is a $L_\pi(C,n)$ if and only if its twisted k-invariant is 0.*

PROOF. Exercise. See Olum [13] or Robinson [17]. The case $n = 1$ is treated in [7].

COROLLARY 2.3. *The universal classifying space for $K(G,1)$ bundles is a $L_{\mathrm{Out}\,G}(C,2)$ iff the universal example $U = 0$.*

3. We prove Theorems C and F and give a geometric proof of Theorem E.

Let $AK(G,1)$ be the semisimplicial group complex of all automorphisms of $K(G,1)$ and let $AK(G,1) \to W \xrightarrow{q'} BAK(G,1) = B$ be its universal principal fibration. Then the associated $K(G,1)$ bundle

$$AK(G,1) \times_{AK(G,1)} K(G,1) \to W \times_{AK(G,1)} K(G,1) \to B$$

is the universal $K(G,1)$ fibration $K(G,1) \to T \xrightarrow{q} B$. Compare with [1, 5.6]. By Gottlieb [6], $\pi_i(B)$ is $\mathrm{Out}\,G$, if $i = 1$; C, if $i = 2$; and 0, otherwise (where C is the center of G). Essentially, since the homomorphism induced on the fundamental groups by the map $AK(G,1) \times K(G,1) \to K(G,1)$ is addition $C \times G \to G$, the homomorphism $\partial\colon \pi_2 \to \pi_1$ in the homotopy sequence for q "is" the inclusion of C into G. Thus T is a $K(\ ,1)$ and the homotopy sequence for q reduces to $0 \to C \to G \to \pi_1(E) \to \mathrm{Out}\,G \to 1$. In §7 we will prove the following:

THEOREM 3.1. *There is a Kan fibration $K(G,1) \xrightarrow{i} E \xrightarrow{p} B$ with the properties:*

1. *The long exact homotopy sequence for p is all zero except for* $0 \to \pi_2(B) \to \pi_1(K(G,1)) \to \pi_1(E) \to \pi_1(B) \to 1$, *and this is* $0 \to C \to G \to \operatorname{Aut} G \to \operatorname{Out} G \to 1$.
2. *The "action" of $\pi_1(B)$ on $\pi_1(K(G,1))$ which is a homomorphism* $\pi_1(B) \to \operatorname{Out} G$ *is the identity.*
3. *The twisted k-invariant for B is $u = \Phi(U)$, as described in Theorem* C.

Since this is a fibration with fiber a $K(G,1)$, there is a map $f\colon B \to BAK(G,1)$ which induces p.

THEOREM 3.2. *The map f is a homotopy equivalence, and the constructed bundle is the universal bundle.*

Theorems C and F easily follow from 3.1 and 3.2.

PROOF OF 3.2. Since actions of π_1 (base) on the fiber are preserved under pullbacks, it follows from (2) that the action in the universal fibration is also the identity and that $f_*\colon \pi_1(B) \to \pi_1(BAK(G,1))$ is an isomorphism. The map of long exact homotopy sequences induced by f now shows f induces an isomorphism in homotopy groups.

GEOMETRIC PROOF OF THEOREM E. A homomorphism $\phi\colon \Pi \to \operatorname{Out} G$ induces a map $\phi'\colon K(\Pi,1) \to K(\operatorname{Out} G,1)$ and using §2 we get the following diagram (where, as usual, we confuse the element $u \in H^3_\phi(K(\operatorname{Out} G,1);\{C\})$ and a map $u\colon K(\operatorname{Out} G,1) \to L_{\operatorname{Aut} C}(C,3)$ such that $u^*(v) = u$ (given by 2.1.2)):

(3.3)
$$\begin{tikzcd}
K(G,1) \arrow[r] & E \arrow[d, "p"] & & K(C,2) \arrow[d] \\
K(C,2) \arrow[r] & B \arrow[d] \arrow[rr] & & T_{\operatorname{Aut} C}(C,3) \arrow[d] \\
K(\Pi,1) \arrow[ur, dashed, "\phi''"] \arrow[r, "\phi'"] & K(\operatorname{Out} G,1) \arrow[rr, "u"] & & L_{\operatorname{Aut} C}(C,3)
\end{tikzcd}$$

Let "the obstruction" be $\phi'^*(u)$. Then, as usual, the obstruction $= 0$ iff there is a lifting ϕ'' of ϕ' (by 2.1.3) iff there is a $K(G,1)$ fibration over $K(\Pi,1)$ with "induced" action ϕ (this last follows since p is the universal fibration, by using pullbacks, and since $\phi = \phi''_*\colon \pi_1(K(\Pi,1)) \to \pi_1(B)$). If the obstruction is zero, i.e., if there is a $K(G,1)$ bundle over $K(\Pi,1)$ with "action" ϕ, then $H^2_\phi(K(\Pi,1);\{C\})$ is in 1-1 correspondence with the set of homotopy classes of liftings $K(\Pi,1) \to B$ of maps homotopic to ϕ' (by 2.1.3) which is the set of homotopy classes of maps $K(\Pi,1) \to B$ which induce the homomorphism $\phi\colon \pi_1(K(\Pi,1)) \to \pi_1(B)$, which in turn is in 1-1 correspondence with the equivalence classes of $K(G,1)$ fibrations over $K(\Pi,1)$ with "action" ϕ.

4. We briefly recall the basic algebra we need.

The bar construction on Π, $\{B_n, \partial_n\}$, is a differential graded Π-module, where B_n is the free Abelian group generated by all symbols of the form $x_0[x_1|\cdots|x_n]$, where $x_i \in \Pi$ and $x_i \neq 1$ if $i \geq 1$. See [9, pp. 114, 189]. If the Abelian group C is a Π-module by $\gamma\colon \Pi \to \operatorname{Aut} C$, then get

$$\partial_n^*: \operatorname{Hom}_{Z(\Pi)}(B_{n-1}, C) \to \operatorname{Hom}_{Z(\Pi)}(B_n, C)$$

and define $H_\gamma^n(K; C) = \ker \partial_{n+1}^* / \operatorname{im} \partial_n^*$.

Let G be a group, C its center and α: $\operatorname{Out} G \to \operatorname{Aut} C$ the natural homomorphism. Let ϕ: $\Pi \to \operatorname{Out} G$ be a homomorphism. Consider the diagram

(4.1)
$$\begin{tikzcd} & & \operatorname{In} G \arrow[dr, "i'"] \arrow[dl, "p'"'] & & \Pi \arrow[dll, dashed, "\phi'"'] \arrow[d, "\phi"] & \\ 0 \arrow[r] & C \arrow[r, "i"] & G \arrow[r, "j"] & \operatorname{Aut} G \arrow[r, "p"] & \operatorname{Out} G \arrow[r] & 1 \end{tikzcd}$$

where p and p' are the quotient homomorphisms, i and i' are the inclusions, and $j = i'p'$ so that the row is exact. For each $x \in \Pi$, pick $\phi'(x) \in \operatorname{Aut} G$ so that $p\phi' = \phi$ (ϕ' is just a function!) but pick $\phi'(1) = 1$. Since $p[\phi'(x)\phi'(y)\phi'(xy)^{-1}] = 1$, for each $(x,y) \in \Pi \times \Pi$ we can pick $g(x,y) \in G$ such that $\phi'(x)\phi'(y) = jg(x,y)\phi'(xy)$, but pick $g(1,y) = 1 = g(x,1)$. The associative law in $\operatorname{Aut} G$ gives $j[\phi'(x)\{g(y,z)\}g(x,yz)] = j[g(x,y)g(xy,z)]$ so that for each x, y, $z \in \Pi$ there is a $K(x,y,z) \in C$ such that

(4.2)
$$\phi'(x)\{g(y,z)\}g(xy,z) = iK(x,y,z)g(x,y)g(xy,z).$$

Let $c \in \operatorname{Hom}_{Z(\Pi)}(B_3(\Pi), C)$ be given by

(4.3)
$$c(x_0[x_1|x_2|x_3]) = \{\alpha\phi(x_0)\} K(x,y,z).$$

Then by Eilenberg and Mac Lane [4] (or see [9, pp. 126–128]):

THEOREM 4.4. *The cochain c is a cocycle and its cohomology class $\{c\} \in H^3_{\alpha\phi}(\Pi; C)$ is independent of the two choices made in the construction of K. There is an extension of G by Π which induces ϕ if and only if $\{c\} = 0$.*

Consider the special case $\Pi = \operatorname{Out} G$ and $\phi = \mathrm{id}$. We denote by

(4.5)
$$\begin{aligned} &v\colon \operatorname{Out} G \to \operatorname{Aut} G, \quad f\colon \operatorname{Out} G \times \operatorname{Out} G \to G, \\ &k\colon \operatorname{Out} G \times \operatorname{Out} G \times \operatorname{Out} G \to C \end{aligned}$$

the functions constructed as above. We let $U \in H^3_\alpha(\operatorname{Out} G; C)$ be the corresponding cohomology class.

THEOREM 4.6. *This class $U \in H^3_\alpha(\operatorname{Out} G; C)$ is the universal example for this obstruction. I.e., given ϕ: $\Pi \to \operatorname{Out} G$, then $\phi^*(U) \in H^3_{\alpha\phi}(\Pi; C)$ is the obstruction to the existence of an extension of G by Π which induces ϕ.*

PROOF. Simply let $\phi' = v\phi$, $g = f(\phi \times \phi)$, and $K = k(\phi \times \phi \times \phi)$, verify the relations, and use 4.4.

5. **Description of the natural isomorphism Φ.** For a group Π, let $K(\Pi)$ be the standard Eilenberg-Mac Lane semisimplicial complex which is a Kan complex and is a $K(\Pi, 1)$. Its n-simplices are ordered n-tuples of elements of Π, $\langle a_1, \ldots, a_n \rangle$ for $n > 0$, and a single 0-simplex $\langle\ \rangle$. It has face and degeneracy operators

$$\partial_0\langle a_1,\dots,a_n\rangle = \langle a_2,\dots,a_n\rangle,\quad \partial_n\langle a_1,\dots,a_n\rangle = \langle a_1,\dots,a_{n-1}\rangle,$$
$$\partial_i\langle a_1,\dots,a_n\rangle = \langle a_1,\dots,a_ia_{i+1},\dots,a_n\rangle,\quad 0<i<n,$$
$$s_i\langle a_1,\dots,a_n\rangle = \langle a_1,\dots,a_i,1,a_{i+1},\dots,a_n\rangle,\quad 0\le i\le n.$$

Let $B(\Pi)$ be the bar consturction on Π (see §4), let C be a Π-module, $c\colon B_n(\Pi)\to C$ an equivariant cocycle so that $\{c\}\in H_\phi^n(\Pi;C)$.

LEMMA 5.1. *The canonical isomorphism* $\Phi\colon H_\phi^*(\Pi;C)\to H_\phi^*(K(\Pi);\{C\})$ *is given by* $\Phi(\{c\})\langle a_1,\dots,a_n\rangle = c(1[a_1|\cdots|a_n])$.

PROOF. Extend the argument given in [7, §3].

6. **A construction of a certain Kan fibration.** If $\alpha\colon\Pi\to G$ is a group homomorphism, then $\alpha'\colon K(\Pi)\to K(G)$ by $\alpha'\langle a_1,\dots,a_n\rangle = \langle \alpha a_1,\dots,\alpha a_n\rangle$ is a simplicial map and $\alpha'_*\colon \pi_1(K(\Pi),\langle\,\rangle)\to\pi_1(K(G),\langle\,\rangle)$ is α.

LEMMA 6.1. *If* $1\to G\xrightarrow{i} E\xrightarrow{p}\Pi\to 1$ *is exact, then* $p'\colon K(E)\to K(\Pi)$ *is a Kan fibration with* $i'\colon K(G)\to K(E)$ *the inclusion of the fiber.*

PROOF. This is well known and is just a checking of the definitions.

We will need an explicit description of $T = K(E)$ as a twisted cartesian product $K(G)\times_\tau K(\Pi)$. The extension in 6.1 induces a homomorphism $\phi\colon\Pi\to\operatorname{Out}G$. Following §4, pick a function $\phi'\colon\Pi\to\operatorname{Aut}G$ such that $p\phi'=\phi$, $\phi'(1)=1$. Since the obstruction is zero, we can pick a $g\colon\Pi\times\Pi\to G$ such that $k(x,y,z)=0$, all $x,y,z\in\Pi$ (see [9, p. 127, 8.5]). Thus

$$\phi'(x)\{g(y,z)\}g(x,yz) = g(x,y)g(xy,z) \tag{6.2}$$

(and $g(x,1)=1=g(1,y)$). Then $T=K(G)\times K(\Pi)$ as a set with the definitions of i', p', s_i for all i, and ∂_i for $i>0$ obvious.

6.3. *Define* $\partial_0(a,b) = (\tau(b)\partial_0 a,\partial_0 b)$, where

$$(\tau(b)\partial_0 a)_i = \phi'(b_1)^{-1}\{g(b_1,b_2\cdots b_{i-1})a_ig(b_1,b_2\cdots b_i)^{-1}\},$$

for $i=2,\dots,n$, $a=\langle a_1,\dots,a_n\rangle\in K(G)_n$, $b=\langle b_1,\dots,b_n\rangle\in K(\Pi)_n$. Easily, with details left to the reader, T is a semisimplicial complex and $K(G)\to T\to K(\Pi)$ is the required Kan fibration. We note that the relation $\partial_0\partial_0=\partial_0\partial_1$ is exactly where 6.2 is needed. (6.3 was obtained from p' by working backwards. See [10, §19].)

7. **Proof of 3.1.** Let U be as in Theorem B and $u=\Phi(U)$ (see 5.1). Let $K(C,2)\to B\xrightarrow{q} K(\operatorname{Out}G)$ be the (pullback) twisted cartesian product with (twisted) k-invariant u (see 3.3). So $B=K(C,2)\times K(\operatorname{Out}G)$ as a set, s_i all i and ∂_i for $i>0$ are clear. The twisting function for B is induced by u from that of $T_{\operatorname{Aut}C}(C,3)$ and hence from that of $L(C,3)$, which we denote by τ_1 and is given in May [10, p. 102 (ii)]. It is not hard to see that

$$\partial_0(w,b) = (\alpha(b_1)^{-1}(\tau_1(b'^*u)+\partial_0 w),\partial_0 b) \tag{7.1}$$

where $b=\langle b_1,\dots,b_n\rangle$, $\alpha\colon\operatorname{Out}G\to\operatorname{Aut}C$ is the natural homomorphism, and $b'\colon \Delta[n] \to K(\operatorname{Out} G)$ is the simplicial map such that $b'(0,1,\dots,n) = b$ (so that $b'^*(u) \in Z^3(\Delta[n]; C)$).

We build E as a triply twisted cartesian product. As a set, $E = K(G) \times B = K(G) \times K(C,2) \times K(\operatorname{Out} G)$. The maps $i\colon K(G) \to E$, $p\colon E \to B$, s_i for all i, and ∂_i for $i > 0$ are all clear.

7.2. *Define* ∂_0 in E by $\partial_0(a,w,b) = (\tau(b)(\sigma(w) + \partial_0 a), \partial_0(w,b))$. Here, $\partial_0(w,b)$ is as in 7.1, σ is the twisting function for the fibration $K(C) \to L(C,2) \xrightarrow{\pi} K(C,2)$, $+\colon K(C) \times K(G) \to K(G)$ is induced by multiplication $C \times G \to G$, and $\tau(b)(\ )$ is essentially the same τ given by (6.3) except that the g and ϕ' are replaced with the f and v of 4.4.

It will now follow that p is a Kan fibration and that E is a Kan complex. Verifying the relation $\partial_0\partial_0 = \partial_0\partial_1$ is a little involved and is exactly where (4.2) is needed. One step is to observe for 3-simplices $(0,1,2,j) \in \Delta[n]$, $b'(0,1,2,j) = \langle b_1, b_2, b_3, \dots, b_j\rangle$, so that

$$\begin{aligned} b'^*(u)(0,1,2,j) &= u\langle b_1, b_2, b_3 \cdots b_j\rangle = U[b_1|b_2|b_3 \cdots b_j] \quad \text{by 5.1} \\ &= k(b_1, b_2, b_3 \cdots b_j) \quad \text{by 4.3.} \end{aligned}$$

Remaining details are left to the reader.

Proof of 3.1.3. Immediate by the construction.

Proof of 3.1.1. By construction, $\pi_1(B) = \operatorname{Out} G$, $\pi_2(B) = C$, and $\pi_i(B) = 0$ otherwise. Using the construction of E, it is easy to construct a map of fibrations from π to p such that $K(C) \to K(G)$ and $K(C,2) \to B$ are both the natural inclusions. Naturality now yields that $\partial\colon \pi_2(B) \to \pi_1(K(G))$ is the inclusion $C \to G$.

The extension $\operatorname{In} G \xrightarrow{i'} \operatorname{Aut} G \xrightarrow{q} \operatorname{Out} G$ induces a $\phi\colon \operatorname{Out} G \to \operatorname{Out}(\operatorname{In} G)$. From diagram (4.1) we get the following diagram

(7.3)
$$\begin{tikzcd}
0 \arrow[r] & C \arrow[r, "i"] & G \arrow[rr, "j"] \arrow[dr, "p'"] & & \mathrm{Aut}\,G \arrow[r, "q"] \arrow[dd, "\beta"] & \mathrm{Out}\,G \arrow[r] \arrow[dd, "\phi"] \arrow[l, bend left, "v"'] \arrow[ddl, dashed, "\phi'"'] & 1 \\
& & & \mathrm{In}\,G \arrow[ur, "i'"] \arrow[dr, "\gamma"] & & & \\
& \mathrm{Out}\,G \times \mathrm{Out}\,G \arrow[uur, "f"] \arrow[urr, dashed, "g"] & & & \mathrm{Aut}(\mathrm{In}\,G) \arrow[r] & \mathrm{Out}(\mathrm{In}\,G) &
\end{tikzcd}$$

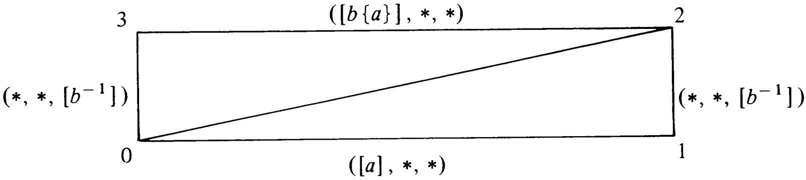
where v and f are given in 4.4. Let $\phi' = \beta v$ and $g = p'f$, and construct the Kan fibration $K(\operatorname{In} G) \to K(\operatorname{Aut} G, 1) \to K(\operatorname{Out} G)$ as in §6. From the explicit twisting functions given, we get a map of fibrations

$$\begin{tikzcd}
K(G) \arrow[r] \arrow[d, "(p')'"] & E \arrow[r, "p"] \arrow[d] & B \arrow[d] \\
K(\mathrm{In}\,G) \arrow[r] & K(\mathrm{Aut}\,G, 1) \arrow[r] & K(\mathrm{Out}\,G)
\end{tikzcd}$$

Using the induced map of homotopy sequences and the fact that $(p')'_* = p'$ on π_1, the rest of 3.1.1 follows.

Proof of 3.1.2. Let $S^1 = \Delta[1]/\{(0),(1)\}$ and identify the 1-cell $[a] \in K(G)$ with the map $S^1 \to K(G)$ generated by $(0,1) \to [a]$. Then the isomorphism $G \cong \pi_1(K(G), *)$ $(* = [1] =$ the only 0-cell$)$ is induced by $a \leftrightarrow [a]$. Similarly for $K(\operatorname{Out} G)$ and for $B = K(C,2) \times_\rho K(\operatorname{Out} G)$. Pick $a \in G$ and b

$\in \operatorname{Out} G$. To prove 3.1.2, it is sufficient to find a homotopy $S^1 \times I \to E$ which starts with $([a], *, *)$, ends with $([b^{-1}\{a\}], *, *)$ and which lies over $(*, [b^{-1}])$ $\in B$. Using May [10, 5.1 and 6.2], it is sufficient to exhibit two 2-cells of E which match up along a common boundary and which satisfy the conditions indicated in the diagram



Letting $\langle 0, 1, 2\rangle = ([a|1], *, [1|b^{-1}])$ and $\langle 0, 2, 3\rangle = ([1|a], *, [b^{-1}|1])$ (note that $\partial_0\langle 0, 2, 3\rangle = ([(b^{-1})^{-1}\{a\}], *, [1]))$ completes the proof.

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