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EMBEDDING CONTRACTIBLE 2-COMPLEXES IN E^4

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ABSTRACT. If L is any figure eight complex or any complex of type $(1,1,1)$, then there are infinitely many different embeddings of L in E^4.

1. **Introduction.** By generalizing Mazur's embedding of the dunce hat in S^4 (refer to [5]), Glaser [2] has constructed infinitely many different contractible 2-complexes each embedded piecewise linearly in S^4 so as to have nonsimply connected complements. Neuzil [4] has constructed an embedding of the dunce hat in S^4 with nonsimply connected complement.

In this paper, we extend Neuzil's result to complexes of type $(1,1,1)$ and do the same for figure eight complexes. Moreover, if L is any one of these complexes, then there are infinitely many distinct embeddings of L in E^4.

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2. **Definitions and notation.** A figure eight complex with sewing words $\alpha^r\beta^s$, $\alpha^m\beta^n$ is a contractible 2-complex obtained by attaching two disks D_1, D_2 to a figure eight $\alpha \vee \beta$ by the formula $\alpha^r\beta^s$, $\alpha^m\beta^n$, respectively, where r, s, m, n are positive integers such that $rn - sm = \pm 1$. This last condition guarantees contractibility.

A complex of type $(1,1,1)$ with sewing word $\alpha^{\epsilon(1)}\alpha^{\epsilon(2)}\cdots\alpha^{\epsilon(p)}$ is a contractible 2-complex obtained by attaching a disk to a circle α by the formula $\alpha^{\epsilon(1)}\alpha^{\epsilon(2)}\cdots\alpha^{\epsilon(p)}$ where $\epsilon(i) = \pm 1$ and $\sum_{i=1}^{p}\epsilon(i) = \pm 1$. This last condition guarantees contractibility. We note that the dunce hat is an example of a complex of type $(1,1,1)$ with sewing word $\alpha^{-1}\alpha^1\alpha^1$.

By a knot group G presented in the usual manner

$$G = \{x_1, x_2, \ldots, x_n | r_1 = r_2 = \cdots = r_m = 1\}$$

we mean the following: for a tame simple closed curve C in E^3 let $G = \Pi_1(E^3 - C)$. We suppose the knot C has a presentation with respect to which it is divided into n arcs by its undercrossing points, and that G is generated by x_1, x_2, $\ldots$, x_n where x_j is represented by a simple closed curve in $E^3 - C$ which encircles once the jth arc and passes under no other arc.

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If M is a manifold then Bd M and Int M will denote the boundary of M and interior of M, respectively. We will use $\cong$ to denote group isomorphism and $\approx$ to denote homeomorphism. By $(m,n) = 1$, we mean that the two integers m and n are relatively prime.

3. Main results.

LEMMA. *Suppose f is a map of X onto Y and $A \subseteq X$. If $f|f^{-1}(Y - f(A))$ is a homeomorphism then $M(f) - M(f|A)$ is homotopically equivalent to $X - A$.*

The proof is by deformation retraction along the fibers of the mapping cylinder.

The first theorem is just a generalization of Neuzil's result in [4].

THEOREM 1. *Suppose G is a knot group and G is presented in the usual manner:*

$$G = \{x_1, x_2, \ldots, x_n | r_1 = r_2 = \cdots = r_m = 1\}.$$

If L is a complex of type (1,1,1) with sewing word $\alpha^{\epsilon(1)}\alpha^{\epsilon(2)}\cdots\alpha^{\epsilon(p)}$, then for any p-tuple of integers $i(1), i(2), \ldots, i(p)$ between 1 and n, there is an embedding of L in E^4 such that $\Pi_1(E^4 - L)$ is presented by

$$\{x_1, x_2, \ldots, x_n | r_1 = r_2 = \cdots = r_m = 1, x_{i(1)}^{\epsilon(1)} x_{i(2)}^{\epsilon(2)} \cdots x_{i(p)}^{\epsilon(p)} = 1\}$$
$$= G/\{x_{i(1)}^{\epsilon(1)} x_{i(2)}^{\epsilon(2)} \cdots x_{i(p)}^{\epsilon(p)}\}.$$

PROOF. Let $G = \Pi_1(E^3 - C)$, where C is a tame simple closed curve in E^3. Let K be an unknotted polyhedral simple closed curve in $E^3 - C$ such that K is in the equivalence class represented by the word $x_{i(1)}^{\epsilon(1)} x_{i(2)}^{\epsilon(2)} \cdots x_{i(p)}^{\epsilon(p)}$ in the given presentation of G where $\epsilon(i) = \pm 1$ and $\sum_{i=1}^{p} \epsilon(i) = \pm 1$. We choose K so that K bounds a tame disk A in E^3 such that $A \cap C$ is exactly p interior points of A.

Let T be a solid 3-dimensional torus in E^3. We write $T = S^1 \times D^2$. Let h be a homeomorphism of E^3 onto itself which maps C into Int T and maps A onto a meridional disk of T. Divide S^1 into two arcs I_1 and I_2 such that

$$(I_2 \times D^2, (I_2 \times D^2) \cap h(C)) \approx (I_2 \times D^2, I_2 \times \{y_1, y_2, \ldots, y_p\}),$$

where $y_1, y_2, \ldots, y_p$ are distinct points in Int D^2. By putting an orientation on I_2, we can pick K and h and orient C so that the induced orientation on $I_2 \times y_i$ is + if $\epsilon(i) = +1$ and − if $\epsilon(i) = -1$.

Let B^3 be a polyhedral 3-cell in E^3 containing T in its interior. Let f be a piecewise linear map of B^3 onto itself which leaves Bd B^3 pointwise fixed, is a homeomorphism on $B^3 - T$, shrinks $I_1 \times D^2$ to a point, and maps $I_2 \times D^2$ to the center core $S^1 \times 0$ of $S^1 \times D^2$. Then the mapping cylinder of f, M_f, is homeomorphic to $B^3 \times [0,1]$. Let $L' = M_{f'}$, where $f' = f|h(C)$. Then $(B^3 \times [0,1], L') \cap (E^3 \times \{t\}) \approx (B^3, h(C))$ if $0 \le t < 1$ and $(B^3 \times [0,1], L') \cap (E^3 \times \{1\}) \approx (B^3, S^1)$. By the Lemma, $(B^3 \times [0,1]) - L'$ is homotopically equivalent to $B^3 - h(C)$.

Let B_1 be a 3-ball in B^3 containing $h(C)$ in its interior. Let g be a piecewise linear map of B^3 onto itself which leaves Bd B^3 pointwise fixed, is a homeomorphism on $B^3 - B_1$, and shrinks B_1 to a point. Then $M_g \approx B^3 \times [-1,0]$. Let $L'' = M_{g'}$, where $g' = g|h(C)$. L'' is a disk and $L = L' \cup L''$ is a complex of type (1,1,1) with sewing word $\alpha_{i(1)}^{\epsilon(1)}\alpha_{i(2)}^{\epsilon(2)} \cdots \alpha_{i(p)}^{\epsilon(p)}$.

$$(B^3 \times [-1,0], L'') \cap (E^3 \times \{t\}) \approx (B^3, h(C)) \quad \text{if } -1 < t \leq 0$$

and

$$(B^3 \times [-1,0], L'') \cap (E^3 \times \{1\}) \approx (B^3, \text{point}).$$

Again by the Lemma, $(B^3 \times [-1,0]) - L''$ is homotopically equivalent to $B^3 - h(C)$. So $(B^3 \times [-1,1]) - L$ is homotopically equivalent to $B^3 - h(C)$ and applying Van Kampen's theorem we have

$$\Pi_1(E^4 - L) \cong G/\{x_{i(1)}^{\epsilon(1)}x_{i(2)}^{\epsilon(2)} \cdots x_{i(p)}^{\epsilon(p)}\}.$$

THEOREM 2. *Suppose G_1 and G_2 are knot groups each presented in the usual manner*:

$$G_1 = \{x_1, x_2, \ldots, x_a | r_1 = r_2 = \cdots = r_b = 1\} \quad \textit{and}$$
$$G_2 = \{y_1, y_2, \ldots, y_c | s_1 = s_2 = \cdots = s_d = 1\}.$$

Let L be a figure eight complex with sewing words $\alpha^r\beta^s$, $\alpha^m\beta^n$. Then for any $(r+s)$-tuple of integers $i(1), \ldots, i(r), i(r+1), \ldots, i(r+s)$ between 1 and a, and for any $(m+n)$-tuple of integers $j(1), \ldots, j(m), j(m+1), \ldots, j(m+n)$ between 1 and c, there is an embedding of L in E^4 such that

$$\Pi_1(E^4 - L) \cong G_1 * G_2/\{x_{i(1)} \cdots x_{i(r)}y_{j(1)} \cdots y_{j(m)},$$
$$x_{i(r+1)} \cdots x_{i(r+s)}y_{j(m+1)} \cdots y_{j(m+n)}\},$$

where the right side is the group obtained by adding the two relations $x_{i(1)} \cdots x_{i(r)}y_{j(1)} \cdots y_{j(m)} = 1$ and $x_{i(r+1)} \cdots x_{i(r+s)}y_{j(m+1)} \cdots y_{j(m+n)} = 1$ to the free product of G_1 and G_2.

PROOF. Let $G_1 = \Pi_1(E^3 - C_1)$ and $G_2 = \Pi_1(E^3 - C_2)$, where C_1, C_2 are two unlinked tame simple closed curves in E^3. Let K_1 and K_2 be unknotted polyhedral simple closed curves in $E^3 - (C_1 \cup C_2)$, such that K_1 is in the equivalence class represented by the word $x_{i(1)} \cdots x_{i(r)}y_{j(1)} \cdots y_{j(m)}$ and K_2 is in the equivalence class represented by the word $x_{i(r+1)} \cdots x_{i(r+s)}y_{j(m+1)} \cdots y_{j(m+n)}$ in $G_1 * G_2 = \Pi_1(E^3 - (C_1 \cup C_2))$. We may choose K_1 and K_2 so that they bound disjoint polyhedral disks A_1 and A_2, respectively, in E^3, where $A_1 \cap C_1, A_1 \cap C_2, A_2 \cap C_1, A_2 \cap C_2$ consist of exactly r, m, s and n points, respectively.

Let T be a solid two-holed 3-dimensional torus in E^3, and let M_1, M_2, M_3, and M_4 be the meridional disks of T as seen in Figure 1. These four disks divide T into three cells W_1, W, and W_2 (see Figure 1). Let h be a homeomorphism of E^3 onto itself which maps C_1 and C_2 into Int T and maps

A_i onto M_i, $i = 1, 2$. Furthermore, h is constructed so that $(W_i, W_i \cap (h(C_1) \cup h(C_2)))$ is homeomorphic to $([0,1] \times D^2, [0,1] \times F_i)$, where D^2 is a disk and F_i is a finite set, $i = 1, 2$. That is, we may assume no knotting or tangling of $h(C_1)$ and $h(C_2)$ occurs in W_1 or W_2.

Let B^3 be a polyhedral 3-cell in E^3 containing T in its interior. Let f be a piecewise linear map of B^3 onto itself which leaves Bd B^3 pointwise fixed, is a homeomorphism on $B^3 - T$, shrinks W to a point, and maps $\overline{T - W}$ onto the center core (a figure eight) of T. Then the mapping cylinder of f, $M_f \approx B^3 \times [0,1]$. Let $L' = M_{f'}$, where $f' = f|h(C_1) \cup h(C_2)$. Then

$$(B^3 \times [0,1], L') \cap (E^3 \times \{t\}) \approx (B^3, h(C_1) \cup h(C_2)) \quad \text{if } 0 \leq t < 1,$$

and

$$(B^3 \times [0,1], L') \cap (E^3 \times \{1\}) \approx (B^3, \text{figure eight}).$$

By the Lemma, $(B^3 \times [0,1]) - L'$ is homotopically equivalent to $B^3 - (h(C_1) \cup h(C_2))$.

Since $h(C_1)$ and $h(C_2)$ are unlinked, there exist disjoint 3-balls B_1, B_2 in B^3 containing $h(C_1)$ and $h(C_2)$, respectively. Let g be a piecewise linear map of B^3 onto itself which leaves Bd B^3 pointwise fixed, is a homeomorphism on $B^3 - (B_1 \cup B_2)$, and shrinks B_1 and B_2 to points. Then $M_g \approx B^3 \times [-1, 0]$. Let $L'' = M_{g'}$, where $g' = g|h(C_1) \cup h(C_2)$. L'' is the union of two disjoint disks and $L = L' \cup L''$ is a figure eight complex with sewing words $\alpha^r\beta^s$, $\alpha^m\beta^n$.

$$(B^3 \times [-1,0], L'') \cap (E^3 \times \{t\}) \approx (B^3, h(C_1) \cup h(C_2)) \quad \text{if } -1 < t \leq 0,$$

and

$$(B^3 \times [-1,0], L'') \cap (E^3 \times \{-1\}) \approx (B^3, \{p, q\}).$$

Again by the Lemma, $(B^3 \times [-1,0]) - L''$ is homotopically equivalent to $B^3 - (h(C_1) \cup h(C_2))$. So $(B^3 \times [-1,1]) - L$ is homotopically equivalent to $B^3 - (h(C_1) \cup h(C_2))$, and applying Van Kampen's theorem we have

$$\Pi_1(E^4 - L) \cong G_1 * G_2/\{x_{i(1)} \cdots x_{i(r)} y_{j(1)} \cdots y_{j(m)},$$
$$x_{i(r+1)} \cdots x_{i(r+s)} y_{j(m+1)} \cdots y_{j(m+n)}\}.$$

THEOREM 3. *Suppose G is a knot group and G is presented in the usual manner*:

$$G = \{x_1, x_2, \ldots, x_a | r_1 = r_2 = \cdots = r_b = 1\}.$$

If L is any complex of type $(1,1,1)$ with sewing word $\alpha^{\epsilon(1)}\alpha^{\epsilon(2)} \cdots \alpha^{\epsilon(p)}$ or L is any figure eight complex with sewing words $\alpha^r\beta^s$, $\alpha^m\beta^n$, then for each triple of integers i, j and k between 1 and a there exists an embedding of L in E^4 such that $\Pi_1(E^4 - L)$ is presented by $\{x_1, x_2, \ldots, x_a | r_1 = r_2 = \cdots = r_b = 1 = x_i^{-1} x_j x_k\}$.

Proof for complexes of type $(1,1,1)$ *with sewing word* $\alpha^{\epsilon(1)}\alpha^{\epsilon(2)}\cdots\alpha^{\epsilon(p)}$. Given any three arbitrary generators x_i, x_j, and x_k (not necessarily distinct) of G, we claim that we can pick generators $x_{i(1)}, x_{i(2)}, \ldots, x_{i(p)}$ of G such that $x_{i(1)}^{\epsilon(1)}x_{i(2)}^{\epsilon(2)}\cdots x_{i(p)}^{\epsilon(p)} = x_i^{-1}x_jx_k$ (or $x_kx_i^{-1}x_j$ or $x_jx_kx_i^{-1}$). Then the conclusion follows immediately from Theorem 1.

The proof is by induction on p, the length of the sewing word. The induction begins with $p = 3$. In this case, L is just the dunce hat and the result was first obtained in [4].

Suppose the result is true for all p' such that $3 \leq p' < p$, and consider a complex L of type $(1,1,1)$ with sewing word $\alpha^{\epsilon(1)}\alpha^{\epsilon(2)}\cdots\alpha^{\epsilon(p)}$. Let j be the greatest integer such that $\epsilon(j)$ has the opposite sign as $\epsilon(p)$, (hence $\epsilon(j+1) = -\epsilon(j)$), and consider the complex L' of type $(1,1,1)$ with sewing word $\alpha^{\epsilon(1)}\alpha^{\epsilon(2)}\cdots\alpha^{\epsilon(j-1)}\alpha^{\epsilon(j+2)}\cdots\alpha^{\epsilon(p)}$. Given x_i, x_j, x_k, there exist by induction on $p' = p-2$, generators $x_{i(1)}, x_{i(2)}, \ldots, x_{i(j-1)}, x_{i(j+2)}, \ldots, x_{i(p)}$ of G such that

$$G/\{x_i^{-1}x_jx_k\} \cong G/\{x_{i(1)}^{\epsilon(1)}x_{i(2)}^{\epsilon(2)}\cdots x_{i(j-1)}^{\epsilon(j-1)}x_{i(j+2)}^{\epsilon(j+2)}\cdots x_{i(p)}^{\epsilon(p)}\}.$$

Therefore, by Theorem 1, letting $x_j = x_{j+1} = x$, there exists an embedding of L in E^4 such that

$$\begin{aligned}\Pi_1(E^4 - L) &\cong G/\{x_{i(1)}^{\epsilon(1)}x_{i(2)}^{\epsilon(2)}\cdots x_{i(j-1)}^{\epsilon(j-1)}x^{\epsilon(j)}x^{-\epsilon(j)}x_{i(j+2)}^{\epsilon(j+2)}\cdots x_{i(p)}^{\epsilon(p)}\}\\ &\cong G/\{x_{i(1)}^{\epsilon(1)}x_{i(2)}^{\epsilon(2)}\cdots x_{i(j-1)}^{\epsilon(j-1)}x_{i(j+2)}^{\epsilon(j+2)}\cdots x_{i(p)}^{\epsilon(p)}\}\\ &\cong G/\{x_i^{-1}x_jx_k\}.\end{aligned}$$

Proof for figure eight complexes L with sewing words $\alpha^r\beta^s$, $\alpha^m\beta^n$. Let C be a knot such that

$$\Pi_1(E^3 - C) \cong G = \{x_1, x_2, \ldots, x_a | r_1 = r_2 = \cdots = r_b = 1\}.$$

Applying Theorem 2, where $C_1 = C$ and C_2 is a trivial knot (that is; $\Pi_1(E^3 - C_2) \cong Z = \{z|\quad\}$), there exists an embedding of L in E^4 such that

$$\Pi_1(E^4 - L) \cong G * Z/\{x_{i(1)}\cdots x_{i(r)}z^m, x_{i(r+1)}\cdots x_{i(r+s)}z^n\}.$$

On the other hand if we take C_1 to be trivial and $C_2 = C$, then there exists an embedding of L in E^4 such that

$$\Pi_1(E^4 - L) \cong Z * G/\{z^r(x_{i(1)}\cdots x_{i(m)}, z^s x_{i(m+1)}\cdots x_{i(m+n)}\}.$$

Hence it suffices to show that given positive integers r, s, m and n such that $rn - sm = \pm 1$ and three arbitrary generators (not necessarily distinct) x_i, x_j, and x_k of $G = \Pi_1(E^3 - C)$, either we can choose generators $x_{i(1)}, x_{i(2)}, \ldots, x_{i(r+s)}$ of G so that

$$G * Z/\{x_{(1)}\cdots x_{i(r)}z^m, x_{i(r+1)}\cdots x_{i(r+s)}z^n\} \cong G/\{x_i^{-1}x_jx_k\}$$

or we can choose generators $x_{i(1)}, x_{i(2)}, \ldots, x_{i(m+n)}$ of G so that

$$Z * G/\{z^r x_{i(1)} \cdots x_{i(m)}, z^s x_{i(m+1)} \cdots x_{i(m+n)}\} \cong G/\{x_i^{-1} x_j x_k\}.$$

Case 1: $r = s$ (or $m = n$). If $r = s$ (or $m = n$) it follows from the fact $rn - sm = \pm 1$ that $r = s = 1$ and $n = m \pm 1$ (or that $m = n = 1$ and $r = s \pm 1$). Thus we must consider one of the following 4-tuples (r, s, m, n) of integers $(1, 1, m, m+1)$, $(1, 1, m-1, m)$, $(s+1, s, 1, 1)$ or $(s-1, s, 1, 1)$. For the first set of integers, we add the relations (1) $zx_i^m = 1$ and (2) $zx_i^{m-1} x_j x_k = 1$ to $Z * G$. By (1) we can eliminate z from the presentation ($z = x_i^{-m}$) and (2) becomes $x_i^{-1} x_j x_k = 1$. For the second set of integers, we add the two relations $zx_i^{m-2} x_j x_k$ and zx_i^{m-1} to $Z * G$. For the third set we add the two relations $x_j x_k x_i^{s-1} z = 1$ and $x_i^s z = 1$ to $G * Z$. For the fourth set we add the two relations $x_i^{s-1} z = 1$, $x_j x_k x_i^{s-2} z = 1$ to $G * Z$.

Case 2: $r < s$ (or $r > s$). We first note that if $r < s$ (or $r > s$) then $m \leq n$ (or $m \geq n$). For if $r < s$ and $m > n$, then $\pm 1 = rn - sm < rn - sn = (r - s)n \leq -1$, a contradiction. Similarly, if $r > s$ and $m < n$, then $\pm 1 = rn - sm > rn - rm = r(n - m) \geq 1$, a contradiction. So by Case 1, we can assume $r < s$ and $m < n$ or $r > s$ and $m > n$.

The proof is by induction on $r + s + m + n$, where either $r < s$ and $m < n$ or $r > s$ and $m > n$. The induction begins with the 4-tuple (r, s, m, n) being either (1,2,1,3) or (2,1,3,1). For the first, we add the two relations $zx_i = 1$ and $z^2 x_i x_j x_k = 1$ to $Z * G$, and for the second we add the two relations $z^2 x_i x_j x_k = 1$ and $zx_i = 1$ to $Z * G$. Next we assume the theorem is true for all t such that $r + s + m + n < t$ and either $r < s$ and $m < n$ or $r > s$ and $m > n$.

Now we assume $r + s + m + n = t$ and either $r < s$ and $m < n$ or $r > s$ and $m > n$. We first suppose $r < s$ and $m < n$. Then we have that (r, s, m, n) is of the form $(r, r + u, m, m + v)$, where $u \geq 1$ and $v \geq 1$. We observe that $rv - um = \pm 1$ (that is, (r, u, m, v) is of the desired form), because $\pm 1 = rn - sm = r(m + v) - (r + u)m = rv - um$. Now $r + u + m + v < r + s + m + n = t$ and either $r = u$ or $m = v$, $r < u$ and $m < v$, or $r > u$ and $m > v$. Hence it follows by Case 1 or the inductive hypothesis that the theorem is true for the 4-tuple (r, u, m, v). Thus either (i) generators $x_{i(1)}, x_{i(2)}, \ldots, x_{i(r+u)}$ of G can be chosen so that

$$G * Z/\{a_1 z^m, b_1 z^v\} \cong G/\{x_i^{-1} x_j x_k\}$$

where $a_1 = x_{i(1)} x_{i(2)} \cdots x_{i(r)}$ and $b_1 = x_{i(r+1)} x_{i(r+2)} \cdots x_{i(r+u)}$, or (ii) generators $x_{i(1)}, x_{i(2)}, \ldots, x_{i(m+v)}$ of G can be chosen so that

$$Z * G/\{z^r a_2, z^u b_2\} \cong G/\{x_i^{-1} x_j x_k\}$$

where $a_2 = x_{i(1)} x_{i(2)} \cdots x_{i(m)}$ and $b_2 = x_{i(m+1)} x_{i(m+2)} \cdots x_{i(m+v)}$. If (i) is true for (r, u, m, v), then for $(r, s, m, n) = (r, r + u, m, m + v)$, we add the two relations $a_1 z^m = 1$ and $b_1 a_1 z^{m+v} = 1$ to $G * Z$, which is the same as adding the two relations $a_1 z^m = 1$ and $b_1 z^v = 1$ to $G * Z$. By (i), this is the same as adding $x_i^{-1} x_j x_k = 1$ to G. If (ii) is true for (r, u, m, v), then for $(r, r + u, m, m + v)$, we add the two relations $z^r a_2 = 1$ and $z^{r+u} a_2 b_2 = 1$ to $Z * G$, which is the same as adding $z^r a_2 = 1$ and $z^u b_2 = 1$. By (ii), this is the same as adding $x_i^{-1} x_j x_k = 1$ to G.

Finally, we suppose $r + s + m + n = t$ and $r > s$ and $m > n$. Then we have that (r,s,m,n) is of the form $(s+f, s, n+g, n)$ where $f \geq 1$ and $g \geq 1$. Since $\pm 1 = rn - sm = (s+f)n - s(n+g) = fn - sg$, then (f,s,g,n) is of the desired form. Also $f + s + g + n < r + s + m + n = t$ and either $f = s$ or $g = n$, $f < s$ and $g < n$, or $f > s$ and $g > n$. So by Case 1 or induction, the theorem is true for (f,s,g,n). Thus we can choose generators of G so that either

(i′) $G * Z/\{a_3 z^g, b_3 z^n\} \cong G/\{x_i^{-1} x_j x_k\}$,
where $a_3 = x_{i(1)} x_{i(2)} \cdots x_{i(f)}$ and $b_3 = x_{i(f+1)} x_{i(f+2)} \cdots x_{i(f+s)}$, or

(ii′) $Z * G/\{z^f a_4, z^s b_4\} \cong G/\{x_i^{-1} x_j x_k\}$,
where $a_4 = x_{i(1)} x_{i(2)} \cdots x_{i(g)}$ and $b_4 = x_{i(g+1)} x_{i(g+2)} \cdots x_{i(g+n)}$.

If (i′) is true for (f,s,g,n) then for $(s+f, s, n+g, n)$, we add the two relations $a_3 b_3 z^{n+g} = 1$ and $b_3 z^n = 1$ to $G * Z$. If (ii′) is true then we add the two relations $z^{s+f} b_4 a_4 = 1$ and $z^s b_4 = 1$ to $Z * G$.

COROLLARY. *If L is any figure eight with sewing words $\alpha^r \beta^s$, $\alpha^m \beta^n$ or any complex of type* (1,1,1) *with sewing word $\alpha^{\epsilon(1)} \alpha^{\epsilon(2)} \cdots \alpha^{\epsilon(p)}$, then there exist infinitely many distinct embeddings of L in E^4.*

PROOF. By Theorem 3, it will suffice to show there exist infinitely many distinct groups $G_{i=1}^{\infty}$ such that by adding the appropriate relation $x_i^{-1} x_j x_k = 1$ to each one, we still obtain infinitely many distinct groups.

We start by considering a torus knot $K_{p,q}$ where p and q are relatively prime positive integers and $p > q$. $\Pi_1(E^3 - K_{p,q})$ has a presentation

$$H = \{a_1, a_2, \ldots, a_p | a_k a_{k+1} \cdots a_{k+q+1} = a_{k+1} a_{k+2} \cdots a_{k+q}, k = 1, 2, \ldots, p\}$$

(see [3, p. 155]). Setting $x = a_1 a_2 \cdots a_q$ and $y = a_1 a_2 \cdots a_p$, we get $H' = \{x, y | x^p = x^q\}$ where $a_k = x^{d(1-k)} y^c x^{dk}$, $k = 1, 2, \ldots, p$ and c and d are integers such that $cp + dq = 1$ (see [3]). Letting $p = 2q + 1$, then $cp + dq = 1$ is satisfied for $c = 1$ and $d = -2$. Then $a_k = x^{2(k-1)} y x^{-2k}$. By Theorem 3 there exists an embedding of L in E^4 such that

$$\Pi_1(E^4 - L) \cong \Pi_1(E^3 - K_{p,q})/\{a_1^{-1} a_{q+1} a_{q+2}\}.$$

Under the presentation H' the relation $a_1 = a_{q+1} a_{q+2}$ becomes

$$yx^{-2} = x^{2q} y x^{-2q-2} x^{2q+2} y x^{-2q-4} \quad \text{or} \quad yx^{2q+2} = x^{2q} y^2.$$

So $\Pi_1(E^4 - L)$ is isomorphic to the group with the following presentation:

$$G'_q = \{x, y | x^{2q+1} = y^q, yx^{2q+2} = x^{2q} y^2\}.$$

The second relation may be written as $yxx^{2q+1} = x^{-1} x^{2q+1} y^2$. Hence, substituting y^q for x^{2q+1} we get:

$$G''_q = \{x, y | x^{2q+1} = y^q, xyx = y^2\}.$$

Thus for each positive integer $q > 2$, there exists an embedding of L in E^4 such that $\Pi_1(E^4 - L)$ is isomorphic to the group with presentation:

$$G_q = \{x, y | x^{2q+1} = y^q, (xy)^2 = y^3\}.$$

Now let $q = 3i$ ($i \geq 1$). By adding the relation $y^3 = 1$, we obtain:

$$\hat{G}_i = \{x, y \mid x^{6i+1} = y^3 = (xy)^2 = 1\}.$$

Bing (see [1, p. 35]) showed these groups have a nontrivial representation in the symmetric group S_{2q+1} by sending $x \to (1, 2, 3, \ldots, 6i + 1)$ and $y \to (6i + 1, 2, 1)(6i, 4, 3) \cdots (4i + 2, 4i, 4i - 1)(4i + 1)$. Thus the groups G_q for $q = 3i$ are all nontrivial.

To complete the proof, it suffices to find infinitely many distinct groups of the form G_q, $q = 3i$. To this end, we show that $G_{q'} \not\cong G_q$ for $q = 3i$ and $q' = 3j$, where $6i + 1$ and $6j + 1$ are prime numbers with $q' > q$.

By above, we have a nontrivial homomorphism of G_q into S_{2q+1}. In order to show G'_q is distinct from G_q, we need only show there is no nontrivial homomorphism of $G_{q'}$ into S_{2q+1}. Suppose otherwise, that is, that we have a homomorphism $\psi\colon G_{q'} \to S_{2q+1}$, where $\psi(x') = \hat{x}$ and $\psi(y') = \hat{y}$ ($\hat{x}, \hat{y} \in S_{2q+1}$), and

$$G_{q'} = \{x', y' \mid x'^{2q'+1} = y'^{q'}, (x'y')^2 = y'^3\}.$$

Since $((2q + 1)!, 2q' + 1) = 1$, there are integers b, t such that $b(2q + 1)! + t(2q' + 1) = 1$. Since $\hat{x}^{2q'+1} = \psi(x'^{2q'+1}) = \psi(y'^{q'}) = \hat{y}^{q'}$, then $\hat{x} = \hat{x}^{b(2q+1)!+t(2q'+1)} = \hat{x}^{t(2q'+1)}$ (since the order of $S_{2q+1} = (2q + 1)!$) $= \hat{y}^{tq'}$. Thus $\hat{x}$ and $\hat{y}$ commute. From this and the fact that $\hat{y}^3 = (\hat{x}\hat{y})^2$, we get $\hat{y} = \hat{x}^2$. That and $\hat{x}^{2q'+1} = \hat{y}^{q'}$ imply $\hat{x} = 1$. Also $\hat{y} = \hat{x}^2 = 1$. Thus ψ is trivial. Since ψ was arbitrary the conclusion follows.

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