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ON MODELS $\equiv_{\infty\omega}$ TO AN UNCOUNTABLE MODEL

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ABSTRACT. It is shown that a model is $\equiv_{\infty\omega}$ to an uncountable model provided there is an uncountable model of its complete theory with respect to some admissible fragment containing a copy of the given model.

It has been known for some time that $\mathrm{Th}_A(M)$, the set of all sentences of L_A true of the structure M, need not characterize M in $L_{\infty\omega}$, even though A is admissible and $M \in A$. In particular, if M is countable, it may not be characterized up to isomorphism. For examples of this phenomenon the reader may consult [7], as well as for any other terminology used, but not explicitly introduced herein. Consequently, it is of interest to determine what properties of a structure can be obtained from its L_A theory, where A is any admissible set containing it as an element, and so from the smallest such admissible set.

The primary purpose of this brief paper is to present a result in this direction. In §2 we will show that if a structure M is an element of a countable admissible set A, then there is an uncountable model $\equiv_{\infty\omega}$ to M iff $\mathrm{Th}_A(M)$ has an uncountable model.

Since in our setting $\mathrm{Th}_A(M)$ will be Σ-definable on A, one should naturally look to the well-known result due independently to Gregory [2] and Ressayre [8] which tells exactly when a Σ-definable theory on a countable admissible set has an uncountable model. As far as we can see, our result does not seem to follow from the statement of the Gregory-Ressayre theorem. However, by looking at Ressayre's proof, one is led to a proof of our result.

Ressayre's proof involves the notion of a Σ_A-saturated model. In our proof, we also, in effect, rely on Σ_A-saturated models, but from a different point of view.

Throughout we deal with admissible sets which may contain urelements, but need not contain ω. The primary reference for admissible sets with urelements is, of course, [1]. Admissible sets will be denoted by A, B, etc. Models will be denoted by M, N, etc., and are always taken to be the structures appropriate to the language L.

As the reader is no doubt aware, $M \equiv_{\infty\omega} N$ means that the models M and N satisfy the same sentences of $L_{\infty\omega}$, the infinitary language closed under arbitrary conjunctions and disjunctions, but only finite quantifiers. $M \equiv_A N$ means that M and N agree on all sentences of $L_{\infty\omega}$ in A, while $M \equiv^\alpha N$ means that M and N agree on all sentences of $L_{\infty\omega}$ of quantifier rank at most α. We use similar notation for notions of elementary submodel— $M \prec_{\infty\omega} N$.

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Harnik, M. Makkai, J.-P. Ressayre, and J. Stavi, as well as the National Research Council of Canada for financial assistance during the summer of 1974, when our work on the subject of this paper began. It should be noted that Theorem 2.1 below can be obtained, though in a less direct fashion, using more general results of Makkai [4].

1. In contrast to the counterexample mentioned in our introductory remarks, the following weak result does hold.

LEMMA 1.1. *Suppose A is admissible and M, $N \in A$. Then*

$$M \equiv_A N \quad \textit{iff} \quad M \equiv_{\infty\omega} N.$$

COROLLARY. *Suppose A is admissible, $M, N \in A$, and M and N are countable. Then*

$$M \equiv_A N \quad \textit{iff} \quad M \textit{ and } N \textit{ are isomorphic}.$$

A proof of Lemma 1.1 in a slightly more general case can be found in [5] or [7].

We will apply Lemma 1.1 to obtain a required uniqueness result for Σ_A-saturated models. The notion of a Σ_A-saturated model is due to Ressayre [7], who used the term Σ-compact. The term Σ_A-saturated was used later by Harnik [3]. The notion we introduce below appears very different from Ressayre's notion, but, as observed by Ressayre [8], is, in fact, equivalent to it for a large class of A. The notion stated below was studied by the present author in [6] without knowledge of Ressayre's work, and the existence theorem proved directly for this notion. Since this notion is the one we actually use in our proof of Theorem 2.1 below, it alone will be introduced. Since we did not give a name to this notion in [6], we borrow the term "Σ_A-saturated", although our notion may be weaker in the case of certain A, though not in the natural setting of Lemma 1.4 below.

DEFINITION 1.2. Let A be a countable admissible set and T a theory in L_A. A model M of T is said to be Σ_A-*saturated* iff M is countable and is an element of some admissible $B \supseteq A$, with the same ordinals as A.

Such a set B is usually referred to as a fattening of A. It should be pointed out that, in general, given two Σ_A-saturated models, there may be no common fattening of A containing both of them.

As mentioned earlier, we have the following existence theorem.

THEOREM 1.3. *Let A be a countable admissible set, and suppose T is a consistent Σ-definable theory in L_A. Then T has a Σ_A-saturated model.*

A proof of the above can be found in [6] or [8], though in [6] we adopted the blanket assumption that A contains ω, which was not necessary for this result. Of course, in the case that $\omega \notin A$, there is a much simpler proof (cf. [1]).

Our immediate interest in Σ_A-saturated models is the following strengthening of Lemma 1.1.

LEMMA 1.4. *Let A be a countable admissible set and T a complete theory in L_A. Suppose T has a model in A. Then T has a unique Σ_A-saturated model up to isomorphism.*

PROOF. Assume $M \in A$ is a model of T. Let N be any other $\boldsymbol{\Sigma}_A$-saturated model of T and suppose B is some fattening of A containing N.

Since $M \in A$, one has in A sentences σ_α, for each ordinal $\alpha \in A$, such that $M \models \sigma_\alpha$, and σ_α is complete for all sentences of quantifier rank $\preceq \alpha$, i.e., if ϕ is a sentence of quantifier rank $\preceq \alpha$, then either $\models \sigma_\alpha \to \phi$ or $\models \sigma_\alpha \to \neg\phi$. Consequently, these sentences σ_α are in T, since it is complete for L_A.

Now, since $N \models T$, we have $M \equiv^\alpha N$ where α is the least ordinal not in A. However, since every sentence of L_B has quantifier rank $\preceq \alpha$, we have $M \equiv_B N$. Finally, since M and N are elements of B, by Lemma 1.1, M and N are isomorphic.

Dropping the countability requirement throughout in Lemma 1.4, we could conclude that all "$\boldsymbol{\Sigma}_A$-saturated" models are $\equiv_{\infty\omega}$.

2. We are now prepared to prove the result mentioned in our introductory remarks.

THEOREM 2.1. *Suppose A is a countable admissible set and $M \in A$. The following five conditions are equivalent.*

(i) *There is an uncountable model N such that $M \equiv_A N$.*

(ii) *There are models N_1 and N_2 such that $N_1 \equiv_A M$ and $N_1 \precneqq_A N_2$.*

(iii) *There is a model N such that $M \precneqq_{\infty\omega} N$ (and consequently, by the downward Lowenheim-Skolem theorem, a countable N isomorphic to M).*

(iv) *There is an uncountable model N such that $M \prec_{\infty\omega} N$.*

(v) *There is an uncountable model N such that $M \equiv_{\infty\omega} N$.*

PROOF. (i) $\Rightarrow$ (ii). For N_2 choose some uncountable model of T, the complete theory of M in L_A. By the downward Lowenheim-Skolem theorem, there is some countable N_1 such that $N_1 \prec_A N_2$. Clearly, $M \equiv_A N$, and N_1 is a proper submodel of N_2.

(ii) $\Rightarrow$ (iii). We introduce a new unary predicate U to form a language L' suitable for describing the elementary submodel condition in (ii). We form a new theory T' in L'_A by adding to T

$$(\exists x)[\neg U(\alpha)], \qquad \phi \to \phi^{(U)}$$

for each formula ϕ of L_A, where $\phi^{(U)}$ denotes the usual relativization of ϕ to the predicate symbol U. It is clear that for any model of T', the reduction to the original language L of the restriction of that model to U is a model of T and a proper L_A-elementary submodel of the reduction of the model to L.

Since T' is obviously $\boldsymbol{\Sigma}$-definable on A, and since by (ii) T' has a model, by Theorem 1.3 above, T' has an $\boldsymbol{\Sigma}_A$-saturated model M'. As above, let N_1 be the reduction to L of the restriction of M' to U, and let N' be the reduction of M' to L. It is clear that both N_1 and N' are models of T, and that they are even $\boldsymbol{\Sigma}_A$-saturated since each is an element of the same fattening of A which contains M'. Consequently, by Lemma 1.4 above, both N_1 and N' are isomorphic to M. In particular then, there is some proper L_A-elementary embedding f of M into N'.

If $m_1, \ldots, m_k$ are any elements of M, then our hypothesis tells us that $(M, m_1, \ldots, m_k) \equiv_A (N', f(m_1), \ldots, f(m_k))$. We may now appeal to Lemma 1.4 once again to conclude that $(M, m_1, \ldots, m_k)$ and $(N', f(m_1), \ldots, f(m_k))$

are isomorphic. Since this isomorphism holds for every sequence of elements $m_1, \ldots, m_k$ of M, we may conclude that f is actually a proper $L_{\infty\omega}$-elementary embedding of M into N'. It is now routine to find a countable model N such that $M \precneqq_{\infty\omega} N$.

(iii) $\Rightarrow$ (iv). Our objective is to define an $L_{\infty\omega}$-elementary chain of countable models $M \precneqq_{\infty\omega} M_1 \precneqq_{\infty\omega} \cdots \precneqq_{\infty\omega} M_\alpha \precneqq_{\infty\omega} \cdots$ for each countable ordinal α. Then, by the Tarski-Vaught theorem, the union of the chain will be an uncountable $L_{\infty\omega}$-elementary extension of M.

One inductively constructs $M_{\beta+1}$ by noting first that M_β is isomorphic to M, and appealing to (iii). At limit stages one takes the union of the previously constructed chain and appeals to the Tarski-Vaught theorem to guarantee that the union is an $L_{\infty\omega}$-extension of each of the preceding models, and, in particular, again isomorphic to M.

(iv) $\Rightarrow$ (v) and (v) $\Rightarrow$ (i) are both immediate. $\square$

In Theorem 2.1, the set A may be taken as the smallest admissible set containing M. In this case, the version of Theorem 1.3 required is easily proved using only the Barwise Compactness Theorem.

As an application of Theorem 2.1, let us consider the case in which the admissible set A contains only the finite ordinals. Then, if $M \in A$, and M is infinite, M must be built from urelements. If we further require that M is a structure for a finite language, we may then just as well assume that the language is composed of pure sets. Hence, no matter how "wide" A may be, L_A will still be $L_{\omega\omega}$, ordinary finitary logic. Since, by the upward Lowenheim-Skolem theorem for finitary logic, every consistent theory in $L_{\omega\omega}$ has an uncountable model, it follows from Theorem 2.1 that M will be $\equiv_{\infty\omega}$ to an uncountable model, and, in fact, is an $\infty\omega$-elementary submodel of an uncountable model. In summary, we have established

COROLLARY. *Suppose A is countable admissible with only finite ordinals, $M \in A$ is a structure for a finite language, and M is infinite. Then there is an uncountable model N such that $M \prec_{\infty\omega} N$.*

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