

ON RINGS WHOSE LEFT MODULES ARE DIRECT SUMS OF FINITELY GENERATED MODULES

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ABSTRACT. The relationship between rings of finite module type and rings whose left modules have decompositions that complement direct summands is examined by proving that the latter are precisely the rings of the title.

A ring Δ with identity is said to be of *finite module (or representation) type* in case it is left artinian and has only finitely many (isomorphism types of) finitely generated indecomposable left modules. Such a ring is also right artinian and has only finitely many (the same number) finitely generated indecomposable right modules (Eisenbud and Griffith [7]). Auslander [2], [3] and Ringel and Tachikawa [17] have proved that every module over a ring of finite module type is a direct sum of finitely generated modules, and Tachikawa [17] has shown that they all have decompositions $M = \bigoplus_A M_\alpha$ that *complement direct summands* in the sense [1] that for each direct summand K of M there is a subset $B \subseteq A$ with $M = K \oplus (\bigoplus_B M_\beta)$.

Chase [6] proved that the rings of the title are left artinian. (See also [8] and [11].) More recently, Auslander [5] has proved that if Δ is finitely generated over its center and each left Δ -module is a direct sum of finitely generated modules, then Δ is of finite module type, and Fuller and Reiten [9] have noted that if every left and every right Δ -module has a decomposition that complements direct summands, then Δ is of finite module type. However, it is still not known whether either of these conditions on its left modules alone forces an arbitrary ring Δ with identity to be of finite module type. Our purpose here is to prove that nevertheless they are equivalent, and to show how their satisfaction depends on the structure of the finitely generated indecomposable left Δ -modules.

We say that a ring R has *enough idempotents* in case there exists orthogonal idempotents $(e_\alpha)_{\alpha \in A}$ in R (called a *complete set of idempotents for R*) such that $R = \bigoplus_A Re_\alpha = \bigoplus_A e_\alpha R$. By an *R -module* we mean an R -module with a spanning set; so " ${}_R M$ is a left R -module" implies $M = RM = \bigoplus_A e_\alpha M$. (Note then that if $1 \in R$, an R -module is just a unital one.) We denote the categories of left and right R -modules by ${}_R \mathfrak{M}$ and \mathfrak{M}_R , and mention that they contain the regular modules ${}_R R$ and R_R , respectively, and all ordinary submodules of their objects.

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Harada [12] and [13] has pointed out that over a ring R with enough idempotents ordinary direct sums and tensor products behave as they do over a ring with identity, and that the projective R -modules are the direct summands of free modules. Moreover, he has observed that Bass's Theorem P is valid for R . I.e., the following are equivalent: (a) every left R -module has a projective cover; (b) R has a complete set of local idempotents and $J(R)$ is left T -nilpotent; (c) every flat left R -module is projective; (d) R has d.c.c. on principal right ideals.

There is a slight difference for direct products though. The (categorical) direct product of $(M_\gamma)_{\gamma \in C}$ in ${}_R\mathfrak{M}$ is not the ordinary cartesian product $\prod_C M_\gamma$, but rather $R \cdot (\prod_C M_\gamma)$, which we denote by $\hat{\prod}_C M_\gamma$. It is easy to see that this is indeed a direct product in ${}_R\mathfrak{M}$, and, using the idempotents $(e_\alpha)_{\alpha \in A}$, that

$$\begin{aligned} \hat{\prod}_C M_\gamma &= \left\{ rx \in \prod_C M_\gamma \mid r \in R, x \in \prod_C M_\gamma \right\} \\ &= \left\{ x \in \prod_C M_\gamma \mid x = ex \text{ for some } e = e^2 \in R \right\}. \end{aligned}$$

Of course, $\oplus_C M_\gamma \leq \hat{\prod}_C M_\gamma$ and the usual injections and projections do their respective jobs. We also write $M^{(C)}$ for the direct sum of card C copies of M , $M^{\hat{C}}$ for the direct product in ${}_R\mathfrak{M}$, and M^C for the cartesian product.

We shall write all homomorphisms on the right, so fg means "first f then g ".

The principal tool that we shall use is a particular ring with enough idempotents. Let Δ be a ring with identity and let ${}_\Delta U = \oplus_A U_\alpha$ be a direct sum of finitely generated submodules $(U_\alpha)_{\alpha \in A}$. Let

$$R = \{r: {}_\Delta U \rightarrow {}_\Delta U \mid U_\alpha r = 0, \text{ a.e.}\}$$

("a.e." means "for all but finitely many $\alpha \in A$ "), and let $(e_\alpha)_{\alpha \in A}$ be the orthogonal set of idempotents in R such that $Ue_\alpha = U_\alpha$ ($\alpha \in A$). We call R *the functor ring of the $(U_\alpha)_{\alpha \in A}$* , because, although it does not really concern us here, the category ${}_R\mathfrak{M}$ (resp., \mathfrak{M}_R) is equivalent to the category of contravariant (covariant) additive functors from the full subcategory ${}_\Delta\mathfrak{U}$ of ${}_\Delta\mathfrak{M}$ with objects $\{U_\alpha \mid \alpha \in A\}$ to the category of abelian groups (e.g., see [10]). In particular, if $(U_\alpha)_{\alpha \in A}$ consists of one isomorphic copy of each finitely generated left Δ -module, R is *the functor ring of the finitely generated left Δ -modules*.

Let ${}_\Delta U = \oplus_A U_\alpha$ (with each U_α finitely generated) and R be as above. Then ${}_\Delta U_R$ is a bimodule and, letting

$$\overline{\text{Hom}}_\Delta(U, M) = \{\varphi: {}_\Delta U \rightarrow {}_\Delta M \mid U_\alpha \varphi = 0, \text{ a.e.}\}$$

and, for each $f: {}_\Delta M \rightarrow {}_\Delta M'$ in ${}_R\mathfrak{M}$,

$$f_* = \overline{\text{Hom}}_\Delta(U, f): \varphi \mapsto \varphi f \quad (\varphi \in \overline{\text{Hom}}(U, M)),$$

we obtain a covariant functor $\overline{\text{Hom}}_\Delta(U, -): {}_\Delta\mathfrak{M} \rightarrow {}_R\mathfrak{M}$ that commutes with direct products ($\overline{\text{Hom}}_\Delta(U, \prod_C M_\gamma) \cong \hat{\prod}_C \overline{\text{Hom}}_\Delta(U, M_\gamma)$) and direct sums (since $\text{Im } \varphi$ is finitely generated for all $\varphi \in \overline{\text{Hom}}_\Delta(U, M)$), and has left adjoint $(U \otimes_R -): {}_R\mathfrak{M} \rightarrow {}_\Delta\mathfrak{M}$. There are natural transformations

$$\nu: U \otimes_R \overline{\text{Hom}}_\Delta(U, -) \rightarrow {}_1\Delta^{\mathfrak{N}}, \quad \eta: {}_1\Delta^{\mathfrak{N}} \rightarrow \overline{\text{Hom}}_\Delta(U, (U \oplus_R -))$$

defined via

$$\nu_M: u \otimes \varphi \mapsto u\varphi, \quad \eta_N: u \mapsto u \otimes n.$$

Moreover, since $\overline{\text{Hom}}_\Delta(U, U) = R$, $\nu_U: U \otimes_R \overline{\text{Hom}}_\Delta(U, U) \rightarrow {}_\Delta U$ and $\eta_{R,R}: R \rightarrow \overline{\text{Hom}}_\Delta(U, (U \otimes_R R))$ are the canonical isomorphisms. Thus, since both functors commute with direct sums, they restrict to inverse category equivalences

$$\overline{\text{Hom}}_\Delta(U, -): \text{Add}({}_\Delta U) \rightarrow \text{Proj}({}_R R), \quad (U \otimes_R -): \text{Proj}({}_R R) \rightarrow \text{Add}({}_\Delta U)$$

where $\text{Add}({}_\Delta U)$ and $\text{Proj}({}_R R)$ are, respectively, the full subcategories of $\Delta^{\mathfrak{N}}$ and ${}_R \mathfrak{N}$ whose objects are the direct summands of direct sums of copies of U and the projective left R -modules.

In the terminology of Auslander [4], a family of homomorphisms is called noetherian in case, for each sequence of nonisomorphisms

$$M_0 \xrightarrow{f_0} M_1 \xrightarrow{f_1} M_2 \rightarrow \dots$$

in the family, there exists an n such that $f_0 f_1 \cdots f_n = 0$. The family is *conoetherian* if it satisfies the obvious dual condition. In [4] he proved that if, over an artinian ring Δ , the families of monomorphisms and of epimorphisms between finitely generated indecomposable left modules are, respectively, noetherian and conoetherian, then Δ is of finite module type (cf. condition (d) in our Theorem below).

With the above notation and conventions, and acknowledging a great debt to the ideas of Auslander [3] and Harada [12], [13], we are now in position to prove the

THEOREM . *Let Δ be a ring with identity and let R be the functor ring of the finitely generated left Δ -modules. Then the following are equivalent:*

- (a) *Every left Δ -module is a direct sum of finitely generated modules.*
- (b) *Every direct product of projective left R -modules is projective.*
- (c) *R is left perfect.*
- (d) *$1 \in \Delta$ is a sum of orthogonal primitive idempotents and the family of homomorphisms between finitely generated indecomposable left Δ -modules is noetherian.*
- (e) *Every left Δ -module has a decomposition that complements direct summands.*

PROOF. (a) \Rightarrow (b). Assume (a). Then $\text{Add}({}_\Delta U) = \Delta^{\mathfrak{N}}$, and $\text{Proj}({}_R R) \approx \text{Add}({}_\Delta U)$ is closed under direct products.

(b) \Rightarrow (c). For this implication, upon noting that for each $a \in R$, $a \cdot (R^C)$ is just the ordinary abelian group direct product $(aR)^C$, one can virtually copy Chase's proof [6, Theorem 3.1] to show (b) implies that R has d.c.c. on principal right ideals.

(c) \Rightarrow (a). Let $M \in \Delta^{\mathfrak{N}}$, let $C = \overline{\text{Hom}}_\Delta(U, M)$ and let $f = \oplus_C \gamma: U^{(C)} \rightarrow M$. Then, since ${}_\Delta \Delta$ is isomorphic to one of the U_α (so ${}_\Delta U$ is a generator), we

have an epimorphism $U^{(C)} \xrightarrow{f} M \rightarrow 0$ such that each $\gamma \in \overline{\text{Hom}}_{\Delta}(U, M)$ factors through f .

$$\begin{array}{ccc} U^{(C)} & \xrightarrow{f} & M \\ \wr_{\gamma} \swarrow & & \searrow \gamma \\ & U & \end{array}$$

So, since here $\gamma = e\gamma = e\gamma f$ for some $e = e^2 \in R$, the sequence

$$\overline{\text{Hom}}_{\Delta}(U, U^{(C)}) \xrightarrow{f_*} \overline{\text{Hom}}_{\Delta}(U, M) \rightarrow 0$$

is exact. But $\overline{\text{Hom}}_{\Delta}(U, M) = \bigcup \{ \overline{\text{Hom}}_{\Delta}(U, L) \mid L \leq M \text{ and } L \text{ is finitely generated} \}$ is a direct limit of projective left R -modules, so the hypothesis (c) implies that f_* splits. Now, since ${}_{\Delta}U$ is a generator implies that each ν_M is epic, we can apply the 5-lemma to

$$\begin{array}{ccccccc} & & 0 & & & & \\ & & \downarrow & & & & \\ U \otimes_R \overline{\text{Hom}}_{\Delta}(U, K) & \rightarrow & U \otimes_R \overline{\text{Hom}}_{\Delta}(U, U^{(C)}) & \rightarrow & U \otimes_R \overline{\text{Hom}}_{\Delta}(U, M) & \rightarrow & 0 \\ \nu_K \downarrow & & \nu_{U^{(C)}} \downarrow & & \downarrow \nu_M & & \\ K & \xrightarrow{\quad f \quad} & U^{(C)} & \xrightarrow{\quad f \quad} & M & \rightarrow & 0 \\ \downarrow & & \downarrow & & \downarrow & & \\ 0 & & 0 & & 0 & & \end{array}$$

(with $K = \text{Ker } f$) to see that ν_M is an isomorphism, and hence that f also splits. Thus $\text{Add}({}_{\Delta}U) = {}_{\Delta}\mathcal{R}$. Finally, since (c) also implies that each $\text{End}({}_{\Delta}U_{\alpha}) \cong e_{\alpha} R e_{\alpha}$ is left perfect (so $U_{\alpha} = V_{\alpha 1} \oplus \cdots \oplus V_{\alpha n}$ with each $\text{End}_{\Delta}(V_{\alpha i})$ local), the Crawley-Jønsson-Warfield theorem [18] shows that every module in $\text{Add}({}_{\Delta}U)$ is a direct sum of finitely generated modules (with local endomorphism rings).

(c) \Rightarrow (d). Assume (c) and suppose we have a sequence of Δ -maps

$$U_{\alpha_0} \xrightarrow{a_0} U_{\alpha_1} \xrightarrow{a_1} U_{\alpha_2} \rightarrow \cdots$$

with no a_i an isomorphism and each U_{α_i} indecomposable. Then $U_{\alpha_i} = U e_{\alpha_i}$ and $a_i = e_{\alpha_i} a_i e_{\alpha_{i+1}}$. Let $J = J(R)$. If $e_{\alpha_i} = e_{\alpha_{i+1}}$ then, since $e_{\alpha_i} R e_{\alpha_i}$ is local, $a_i \in J(e_{\alpha_i} R e_{\alpha_i}) = e_{\alpha_i} J e_{\alpha_i}$. If $e_{\alpha_i} \neq e_{\alpha_{i+1}}$ then, since U_{α_i} corresponds to $R e_{\alpha_i}$ under the equivalence $\overline{\text{Hom}}_{\Delta}(U, -): \text{Add}({}_{\Delta}U) \rightarrow \text{Proj}({}_R R)$, $R e_{\alpha_i} \cong R e_{\alpha_{i+1}}$ and, hence, $a_i \in e_{\alpha_i} R e_{\alpha_{i+1}} \subseteq e_{\alpha_i} J e_{\alpha_{i+1}}$. Thus, since J is left T -nilpotent, there is an n such that $a_0 a_1 \cdots a_n = 0$.

(d) \Rightarrow (c). Assume (d). Let ${}_{\Delta}M$ be finitely generated and indecomposable. Let $s \in \text{End}({}_{\Delta}M)$. Then the noetherian condition implies that s is either invertible or nilpotent, so $\text{End}({}_{\Delta}M)$ is local. In particular, $\Delta = \Delta g_1 \oplus \cdots \oplus \Delta g_n$ with the g_i local orthogonal idempotents, so Δ is semiperfect. This implies that each of the finitely generated U_{α} is a direct sum of indecomposable modules which, as we have just seen, must have local endomorphism rings. Thus we can find a complete set of local idempotents $(f_{\alpha})_{\alpha \in A}$ for R (here each

e_α is a sum of some of the f_α 's). If a_0, a_1, \dots is a sequence in $J(R)$ then there exist idempotents f_0, f_1, f_2, \dots (each a sum of f_α 's) with

$$a_0 a_1 \cdots a_n = f_0 a_0 f_1 a_1 f_2 \cdots f_n a_n f_{n+1}.$$

If none of these products is zero we can apply the König graph theorem to obtain a sequence $f_{\alpha_0}, f_{\alpha_1}, \dots$ with none of the products

$$f_{\alpha_0} a_0 f_{\alpha_1} a_1 \cdots f_{\alpha_n} a_n f_{\alpha_{n+1}} = 0.$$

But this is contrary to the noetherian assumption because, since they belong to $J(R)$, none of the $f_{\alpha_i} a_i f_{\alpha_{i+1}}: Uf_{\alpha_i} \rightarrow Uf_{\alpha_{i+1}}$ is invertible.

(a) and (c) \Rightarrow (e). If (a) holds then ${}_\Delta \mathcal{N} = \text{Add}({}_\Delta U) \approx \text{Proj}({}_R R)$. Since the argument (a) \Rightarrow (b) in [1, Theorem 6] clearly is valid for a ring with enough idempotents, each module in $\text{Proj}({}_R R)$ (and hence in ${}_\Delta \mathcal{N}$) has a decomposition that complements direct summands if R is left perfect.

(e) \Rightarrow (d). This is by [15, Lemma 9].

From this Theorem and Auslander's theorem [5] that an artin algebra whose left modules are direct sums of finitely generated modules is one of finite module type, we have (with necessity due to Tachikawa [17]) the

COROLLARY . *If Δ is finitely generated over its center, then Δ is of finite module type if and only if each left Δ -module has a decomposition that complements direct summands.*

Together with [9], the Theorem yields the following result (which was already known to Auslander and to Gruson and Jensen).

COROLLARY . *If every left and every right Δ -module is a direct sum of finitely generated modules then Δ is a ring of finite module type.*

REMARK. (1) Let $(V_\beta)_{\beta \in B}$ represent one copy of each finitely generated indecomposable left Δ -module. If we make the additional assumption that $1 \in \Delta$ is a sum of orthogonal primitive idempotents (so that $\bigoplus_B V_\beta$ is a generator) then we can replace R in conditions (b) and (c) of the Theorem with the functor ring R' of the $(V_\beta)_{\beta \in B}$.

(2) "Every left Δ -module" in condition (e) can be replaced by "every countably generated left Δ -module" or by "every countable direct sum of finitely generated left Δ -modules."

(3) In [2], [4] (and [16]) there has been established a 1-1 correspondence between the rings of finite module type and the Morita equivalence classes of artinian (semiprimary) rings that have global dimension ≤ 2 and dominant dimension ≥ 2 . The correspondence is $\Delta \leftrightarrow R' = \text{End}(V_1 \oplus \cdots \oplus V_n)$ where V_1, \dots, V_n represent the indecomposable left Δ -modules. It seems likely, especially in view of the results of [12]–[14] that an appropriate extension of this correspondence might help to determine whether or not rings whose left modules are direct sums of finitely generated modules need be rings of finite module type. (Are they even right artinian?)

(4) Auslander [5] has actually shown that if every indecomposable left module over an artin algebra is finitely generated, the algebra is of finite

module type. His proof of this theorem and the results of [4] show that the conditions: (a') Δ is left artinian and every indecomposable left Δ -module is finitely generated; and (d') Δ is left artinian and has no infinite sequences $M_0 \rightarrow M_1 \rightarrow M_2 \rightarrow \cdots$ of proper monomorphisms between finitely generated indecomposable left modules, also serve to characterize the rings of the title.

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