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A CHARACTERIZATION OF B-SLOWLY VARYING FUNCTIONS

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ABSTRACT. A measurable function $\varphi > 0$ that satisfies the limit condition $\lim_{x\to\infty}(\varphi(x + t\varphi(x))/\varphi(x)) = 1$ for all t is said to be B-slowly varying. If φ is continuous, this limit is shown to hold uniformly for t in compact sets, and an integral representation is derived.

Recent papers by Moh [5] and Peterson [6] deal with a generalization, due to Beurling, of Wiener's Tauberian Theorem.

THEOREM 1. *Let φ be a positive function satisfying*

$$\lim_{x\to\infty} \frac{\varphi(x + t\varphi(x))}{\varphi(x)} = 1 \quad \text{for each fixed } t, \tag{1}$$

and

$$\lim_{x\to\infty} \frac{\varphi(x)}{x} = 0. \tag{2}$$

Let $K \in L^1$ satisfy $\int_{-\infty}^{\infty} K(t)e^{-i\lambda t}\,dt \neq 0$ for each real λ, and let $f \in L^\infty$. If there exists a constant A such that

$$\lim_{x\to\infty} \int_{-\infty}^{\infty} f(t)K\left[\frac{x-t}{\varphi(x)}\right]\frac{dt}{\varphi(x)} = A\int_{-\infty}^{\infty} K(t)\,dt, \tag{3}$$

then

$$\lim_{x\to\infty} \int_{-\infty}^{\infty} f(t)H\left[\frac{x-t}{\varphi(x)}\right]\frac{dt}{\varphi(x)} = A\int_{-\infty}^{\infty} H(t)\,dt \quad \text{for all } H \in L^1. \tag{4}$$

Wiener's Theorem is the special case, $\varphi \equiv 1$, of Theorem 1.

DEFINITION. A positive, measurable function φ that satisfies (1) is B-slowly varying, and B denotes the class of all such functions. If φ satisfies (1) uniformly for t in every bounded interval (a,b), then φ is uniformly B-slowly varying, denoted $\varphi \in B_u$.

THEOREM 2. *If $\varphi \in B$ is continuous, then $\varphi \in B_u$.*

PROOF. We prove this for t between zero and one. The argument for an arbitrary interval (a,b) follows similarly.

Suppose that φ is not uniformly B-slowly varying. Then there is an $\epsilon \in (0,1)$ and sequences $\{t_n\} \subset (0,1)$ and $\{x_n\}$ tending to infinity, such that

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$$(5)\qquad |\varphi(x_n + t_n\varphi(x_n))/\varphi(x_n) - 1| \geqslant \epsilon \qquad (n = 1, 2, \ldots).$$

The function $f_n(t) = \varphi(x_n + t\varphi(x_n))/\varphi(x_n) - 1$ is continuous and $\lim_{n\to\infty} |f_n(t)| = 0$ for fixed t. So there is an integer N and a sequence $\{\lambda_n\} \subset (0, 1)$ such that

$$(6)\qquad |\varphi(y_n)/\varphi(x_n) - 1| = \epsilon \qquad (n \geqslant N),$$

where $y_n = x_n + \lambda_n\varphi(x_n)$. Set

$$V_n = \{\lambda \in (0, 2+\epsilon): |f_n(\lambda)| < \epsilon/2\},$$

$$W_n = \{\mu \in (0,1): |\varphi(y_n + \mu\varphi(y_n))/\varphi(y_n) - 1| < \epsilon/2(1+\epsilon)\},$$

$$W'_n = \{\lambda = \lambda_n + \mu\varphi(y_n)/\varphi(x_n): \mu \in W_n\}.$$

These sets are Lebesgue measurable, with

$$(7)\qquad \lim_{n\to\infty} \mathfrak{M}(V_n) = 2 + \epsilon, \quad \lim_{n\to\infty} \mathfrak{M}(W_n) = 1.$$

(Korevaar, Van Aardenne-Ehrenfest, DeBruijn [4] cite De Le Vallee Poussin [7] for this. One may also apply Egoroff's Theorem to $f_n(t)$.) $W'_n \subset (0, 2+\epsilon)$, and $\mathfrak{M}(W'_n) \geqslant (1-\epsilon)\mathfrak{M}(W_n)$ so that

$$(8)\qquad \liminf_{n\to\infty} \mathfrak{M}(W'_n) \geqslant 1 - \epsilon.$$

For $\lambda \in W'_n$,

$$(9)\qquad \left|\frac{\varphi(x_n + \lambda\varphi(x_n))}{\varphi(x_n)} - \frac{\varphi(y_n)}{\varphi(x_n)}\right| = \left|\frac{\varphi(y_n)}{\varphi(x_n)}\right| \cdot \left|\frac{\varphi(y_n + \mu\varphi(y_n))}{\varphi(y_n)} - 1\right| < \frac{\epsilon}{2}$$

so that

$$(10)\qquad |\varphi(x_n + \lambda\varphi(x_n))/\varphi(x_n) - 1| > \epsilon/2$$

and, in particular, $\lambda \notin V_n$. Thus $V_n \cap W'_n = \varnothing$, $V_n, W'_n \subset (0, 2+\epsilon)$, so that

$$(11)\qquad 2 + \epsilon \geqslant \liminf_n \mathfrak{M}(V_n \cup W'_n) \geqslant \liminf_n (\mathfrak{M}(V_n) + \mathfrak{M}(W'_n)) \geqslant 3,$$

or $\epsilon \geqslant 1$, which is impossible.

Slowly varying functions.

DEFINITION. A positive, measurable function g is slowly varying if it satisfies the limit condition

$$(12)\qquad \lim_{x\to\infty} \frac{g(x+t)}{g(x)} = 1 \quad \text{for each fixed } t.$$

Let K be the class of all slowly varying functions.

For our purposes, K serves as an analogue to B and motivates much of our work.

THEOREM 3. *Let g be a slowly varying function. Then g satisfies* (12) *uniformly for t in bounded intervals, and there exist functions $c(x)$ and $\epsilon(x)$, ϵ*

continuous, $\lim_{x\to\infty} c(x) = c \in (0, \infty)$, $\lim_{x\to\infty} \epsilon(x) = 0$ *such that*

$$g(x) = c(x)\exp\int_0^x \epsilon(s)\,ds. \tag{13}$$

Karamata [3] proved this theorem for g continuous. More recent work has weakened the hypothesis from continuity to measurability (see, for example, [1], [2], and [4]).

Relation between K and B_u. There is a similarity between the class of uniformly B-slowly varying functions and the class K, for a function $\varphi \in B_u$ that is bounded away from zero on an appropriately chosen sequence is slowly varying:

THEOREM 4. *Let $\varphi \in B_u$. If there is a sequence $\{x_n\} \to \infty$, and constants m, M, δ greater than zero such that*

(i) $m \leqslant x_{n+1} - x_n \leqslant M$ $(n = 1, 2, \ldots)$,
(ii) $\varphi(x_n) \geqslant \delta$ $(n = 1, 2, \ldots)$

then $\varphi \in K$.

PROOF. For each $n = 1, 2, \ldots$, define the function $p_n(x)$ for $x \in [x_n, x_{n+1}]$ by

$$p_n(x) = \frac{1}{2}\left(1 + \sin\left(\frac{\pi}{2}\frac{(2x - x_{n+1} - x_n)}{x_{n+1} - x_n}\right)\right). \tag{14}$$

Then $0 \leqslant p_n(x) \leqslant 1$, $p_n(x_n) = 0$ and $p_n(x_{n+1}) = 1$. Moreover, the functions p_n are continuously differentiable,

$$0 \leqslant p_n'(x) \leqslant \frac{\pi}{2(x_{n+1} - x_n)} \leqslant \frac{\pi}{2m}, \quad \text{and} \quad p_n'(x_n) = p_n'(x_{n+1}) = 0.$$

Set $f = \log \varphi$ and

$$f_1(x) = f(x_n) + [f(x_{n+1}) - f(x_n)]p_n(x) \qquad (x_n \leqslant x \leqslant x_{n+1}). \tag{15}$$

The function f_1 is defined for all $x \geqslant x_1$, it is continuously differentiable, and satisfies the estimates

$$|f'(x)| \leqslant (\pi/m)|f(x_{n+1}) - f(x_n)|$$

and

$$|f_1(x) - f(x)| \leqslant |f(x) - f(x_n)| + |f(x_{n+1}) - f(x_n)|$$

where $x_n \leqslant x \leqslant x_{n+1}$. Thus f_1' and $f_1 - f$ tend to zero as x tends to infinity provided

$$\lim_{x\to\infty} |f(x) - f(x_n)| = 0 \qquad (x_n \leqslant x \leqslant x_{n+1}). \tag{16}$$

For $x_n \leqslant x \leqslant x_{n+1}$, there is a $t \in [0, M/\delta]$ such that $x = x_n + t\varphi(x_n)$. Then

$$|f(x) - f(x_n)| = \left|\log \frac{\varphi(x_n + t\varphi(x_n))}{\varphi(x_n)}\right| \to 0, \tag{17}$$

which gives (16). Choose $\epsilon(x)$ between zero and x, so that ϵ is continuous, $\epsilon(x_1) = f_1'(x_1)$, and $\int_0^{x_1} \epsilon(s)\,ds = f_1(x_1)$. Set

$$\epsilon(x) = f_1'(x) \qquad (x \geqslant x_1), \quad c(x) = \varphi(x)\exp\left(-\int_0^x \epsilon(s)\,ds\right). \tag{18}$$

Then for $x \geqslant x_1$, $c(x) = \exp[f(x) - f_1(x)] \to 1$, and φ satisfies (13). It is a simple matter to verify that such functions are elements of K.

An integral representation. Motivating examples for the class of B-slowly varying functions are functions such as x^p $(p < 1)$ and e^{-x}. The derivatives of these functions tend to zero. This does not hold in general, as with

$$\varphi(x) = \begin{cases} 1 + \sin(1), & 0 \leqslant x \leqslant 1, \\ x^{1/2}(1 + x^{-1/4}\sin x), & x \geqslant 1. \end{cases} \tag{19}$$

Here $\varphi \in B_u$, but $\limsup_{x\to\infty} \varphi'(x) \neq 0$.

But these examples suggest an analogue for the class B of Karamata's representation (13).

THEOREM 5. *Let $\varphi \in B_u$. Then there are functions $c(x)$ and $\epsilon(x)$, ϵ continuous, $0 < c = \lim_{x\to\infty} c(x) < \infty$, and $\lim_{x\to\infty} \epsilon(x) = 0$, such that*

$$\varphi(x) = c(x)\int_0^x \epsilon(s)\,ds. \tag{20}$$

Conversely, if a positive, measurable function φ has the representation (20) with ϵ continuous, tending to zero and $c(x)$ tending to a positive limit, then $\varphi \in B_u$.

In particular, any $\varphi \in B_u$ satisfies (2).

The form of (20) was conjectured by Daniel Shea.

As an example, the function φ given by (19) satisfies (20) with $c(x) = 1 + x^{-1/4}\sin x$, $\epsilon(x) = \frac{1}{2}x^{-1/2}$.

We require some additional machinery before proving the theorem. Define inductively at x a sequence $\{x_n\}$ by

$$x_0 = x, \quad x_n = x_{n-1} + \varphi(x_{n-1}). \tag{21}$$

For $\varphi \in B_u$, this sequence virtually characterizes the behavior of φ provided the x_n become infinite.

LEMMA . *Let $\varphi \in B_u$. Then there is an $\tilde{x}$ such that for any $x \geqslant \tilde{x}$ and the sequence $\{x_n\}$ defined for x by (21), $\lim_{n\to\infty} x_n = \infty$.*

PROOF. Choose $\tilde{x}$ so that for $x \geqslant \tilde{x}$, $t \in [-1, 1]$,

$$\varphi(x + t\varphi(x)) \geqslant \tfrac{1}{2}\varphi(x). \tag{22}$$

Suppose the Lemma is false. Then there is an $x \geqslant \tilde{x}$ with $\{x_n\}$ given by (21) such that $\lim_{n\to\infty} x_n = p < \infty$. This limit exists, of course, since the sequence $\{x_n\}$ increases monotonically. Now, $x_n = x + \sum_{k=1}^n \varphi(x_{k-1})$. The series

$\sum \varphi(x_k)$ converges and, in particular, $\lim_{n\to\infty} \varphi(x_n) = 0$. Now, $p \geqslant \tilde{x}$, so by (22),

$$\varphi(y) \geqslant \tfrac{1}{2}\varphi(p) \qquad (p - \varphi(p) \leqslant y \leqslant p + \varphi(p)). \tag{23}$$

Thus

$$0 = \lim_{n\to\infty} \varphi(x_n) \geqslant \liminf_{y\to p} \varphi(y) \geqslant \tfrac{1}{2}\varphi(p), \tag{24}$$

which contradicts the positivity of φ.

PROOF OF THEOREM 5. Let $\tilde{x}$ be as in the Lemma and define $\{x_n\}$ by (21) for some $x_0 \geqslant \tilde{x}$. Then $\lim_{n\to\infty} x_n = \infty$. Set

$$p_n(x) = \frac{\varphi(x_n)}{2}\left(1 + \sin\left[\frac{\pi}{2\varphi(x_n)}(2x - 2x_n - \varphi(x_n))\right]\right) \qquad (x_n \leqslant x \leqslant x_{n+1}). \tag{25}$$

p_n is continuously differentiable on $[x_n, x_{n+1}]$.

$$0 \leqslant p_n(x) \leqslant \varphi(x_n), \quad p_n(x_n) = 0, \quad p_n(x_{n+1}) = \varphi(x_n),$$

$$0 \leqslant p_n'(x) \leqslant \pi/2, \quad \text{and} \quad p_n'(x_n) = p_n'(x_{n+1}) = 0.$$

Set

$$f(x) = \varphi(x_n) + p_n(x)[(\varphi(x_{n+1}) - \varphi(x_n))/\varphi(x_n)] \qquad (x_n \leqslant x \leqslant x_{n+1}). \tag{26}$$

Then f is continuously differentiable, and

$$|f'(x)| \leqslant (\pi/2)\,|\varphi(x_n + \varphi(x_n))/\varphi(x_n) - 1| \to 0 \quad \text{as } x \to \infty. \tag{27}$$

Finally, for $x \in [x_n, x_{n+1}]$, there is a $t \in [0,1]$ such that $x = x_n + t\varphi(x_n)$. Thus

$$\begin{aligned} \frac{\varphi(x_n)}{\varphi(x_n + t\varphi(x_n))}\left[1 - \left|\frac{\varphi(x_n + \varphi(x_n))}{\varphi(x_n)} - 1\right|\right] &\leqslant \frac{f(x)}{\varphi(x)} \\ &\leqslant \frac{\varphi(x_n)}{\varphi(x_n + t\varphi(x_n))}\left[1 + \left|\frac{\varphi(x_n + \varphi(x_n))}{\varphi(x_n)} - 1\right|\right]. \end{aligned} \tag{28}$$

Hence, $\lim_{x\to\infty} f(x)/\varphi(x) = 1$. Define $\epsilon(x)$ between zero and x_0 so that ϵ is continuous, $\epsilon(x_0) = f'(x_0)$, and $\int_0^{x_0} \epsilon(s)\,ds = f(x_0)$. For $x \geqslant x_0$, set $\epsilon(x) = f'(x)$. Assume further that $\epsilon(x) > 0$ for $x \in [0, x_0]$. Set $c(x) = \varphi(x)/\int_0^x \epsilon(s)\,ds$. Then (20) holds, ϵ is continuous, and $\lim_{x\to\infty} \epsilon(x) = 0$. We need only show that $c(x)$ tends to a finite positive limit. But for $x \geqslant x_0$, $c(x) = \varphi(x)/f(x)$, which tends to one. The converse is easily verified.

Remarks. 1. Measurability of a slowly varying function g is sufficient to assure that the limit condition in (12) holds uniformly for t in finite intervals. An open problem at present is whether or not the hypothesis for Theorem 2 can be similarly weakened.

The contradiction in the proof of Theorem 2 was obtained by constructing sets W'_n of measure bounded below and all contained in a finite interval.

Since

$$\mathfrak{M}(W'_n) \geqslant \inf_n (\varphi(y_n)/\varphi(x_n)), \quad \text{and} \quad W'_n \subset (0, 1 + \sup_n (\varphi(y_n)/\varphi(x_n))),$$

it is impossible for $\varphi \in B$ to have constants m, M, $\delta > 0$ and sequences $\{t_n\} \subset (0,1)$, $\{x_n\} \to \infty$, such that $|\varphi(y_n)/\varphi(x_n) - 1| \geqslant \delta$ and $m \leqslant \varphi(y_n)/\varphi(x_n) \leqslant M$, where $y_n = x_n + t_n\varphi(x_n)$. For continuous functions, the second of these inequalities follows from the first. Measurable functions are not as simple, however. Consider, for example,

$$f(x,t) = \begin{cases} 1/x, & x \geqslant t > 0, \\ 0, & 0 < x < t. \end{cases} \tag{29}$$

For each fixed $t > 0$, $\lim_{x\to 0} f(x,t) = 0$, but for any $\epsilon > 0$, if we choose sequences $\{x_n\} \to 0$, $\{t_n\}$ so that $f(x_n, t_n) \geqslant \epsilon$ $(n = 1, 2, 3, \dots)$, then $\lim_{n\to\infty} f(x_n, t_n) = \infty$.

Let $\delta_n = \varphi(y_n)/\varphi(x_n)$, and suppose $\lim_{n\to\infty}\delta_n = \infty$. What can be said about the sequence $\{\delta_n\}$ relative to φ?

Fix $t \in [1,2]$ and let k be an integer, $k \geqslant 4$. Let $\epsilon \in (0, 2^{1/k} - 1)$ be given, and set

$$V_n(t) = \{\mu \in (t_n, t_n + t) \colon |\varphi(x_n + \mu\varphi(x_n))/\varphi(x_n) - 1| < \epsilon\},$$
$$Q_n(t) = \{\lambda \in [0, 2t] \colon |\varphi(y_n + \lambda\varphi(y_n))/\varphi(y_n) - 1| > 1/4\}.$$

Then

$$\lim_{n\to\infty} \mathfrak{M}(V_n(t)) = t, \quad \lim_{n\to\infty} \mathfrak{M}(Q_n(t)) = 0. \tag{30}$$

There is an $\tilde{x}$ such that for $x \geqslant \tilde{x}$,

$$|\varphi(x + t\varphi(x))/\varphi(x) - 1| < \epsilon. \tag{31}$$

Suppose that $\delta_n \in [k-1, k]$ for some $x_n \geqslant x$. For $\lambda_1 \in V_n(t)$, define λ_j by

$$\lambda_j = \lambda_{j-1} + t\varphi(x_n + \lambda_{j-1}\varphi(x_n))/\varphi(x_n) \qquad (2 \leqslant j \leqslant k). \tag{32}$$

Then

$$\begin{aligned} \frac{\varphi(y_n + ((\lambda_j - t_n)/\delta_n)\varphi(y_n))}{\varphi(y_n)} &= \frac{\varphi(x_n + \lambda_j\varphi(x_n))}{\delta_n\varphi(x_n)} \\ &< (1+\epsilon)^j\delta_n^{-1} \leqslant (1+\epsilon)^k\delta_n^{-1} \leqslant 2/(k-1). \end{aligned} \tag{33}$$

Set $Q'_n = \{(\lambda_j - t_n)/\delta_n \colon \lambda_1 \in V_n(t), \lambda_j \text{ defined by (32)}\}$. Then

$$\mathfrak{M}(Q'_n) \geqslant \inf \frac{(\lambda_j - \lambda_{j-1})(k-1)\mathfrak{M}(V_n)}{\delta_n} \geqslant \frac{t(k-1)(1-\epsilon)}{k}\mathfrak{M}(V_n(t)). \tag{34}$$

But $Q'_n \subset Q_n(t)$ for all $n = 1, 2, \dots$, which contradicts (30).

Thus, given $\{t_n\} \subset (0, 1)$, $\{x_n\} \to \infty$ such that $\delta_n = \varphi(x_n + t_n\varphi(x_n))/\varphi(x_n) \to \infty$, let $k(n)$ be the least integer greater than δ_n, $-[-\delta_n]$. We conclude that

$$(35)\quad \left\{t \in [1,2] : \left|\frac{\varphi(x + t\varphi(x))}{\varphi(x)} - 1\right| < 2^{1/k(n)} - 1;\ \text{for all } x \geqslant x_n\right\} = \varnothing,$$

and we have a uniform bound for the rate of convergence in the *B*-slowly varying limit. A similar argument is applicable when $\lim_{n\to\infty} \delta_n = 0$.

2. Theorem 4 considers uniformly *B*-slowly varying functions bounded away from zero on appropriate sequences. Suppose we add a positive constant to an element in B_u. What can we conclude about this translate?

THEOREM 6. *Let $\varphi \in B_u$, $T(x) = \varphi(x) + \epsilon$, $\epsilon > 0$. Then $T \in B_u \cap K$.*

PROOF. Fix t. Then

$$(36)\qquad \lim_{x\to\infty} \frac{T(x - \epsilon t + tT(x))}{T(x)} = \lim_{x\to\infty} \frac{\epsilon + \varphi(x + t\varphi(x))}{\epsilon + \varphi(x)} = 1,$$

and this limit holds uniformly for t in finite intervals. In the proof of Theorem 4, set $x_n = n$, and $f(x) = \log[T(x - \epsilon(x - n)/T(n))]$ $(n \leqslant x < n + 1)$. Define f by (15) and let $t = (x - n)/T(n)$ for $x \in [n, n + 1]$. Then $0 \leqslant t \leqslant 1/T(n) \leqslant 1/\epsilon$, so for $x \leqslant [n, n + 1]$,

$$(37)\quad |f(x) - f(n)| = \left|\log \frac{T(x - \epsilon t)}{T(n)}\right| = \left|\log \frac{T(n - \epsilon t + \epsilon T(n))}{T(n)}\right| \to 0.$$

So (16) holds, and $T(x - \epsilon(x - n)/T(n)) \in K$, with

$$(38)\qquad T\left(x - \epsilon\frac{x - n}{T(n)}\right) = c(x)\exp\int_1^x \epsilon(s)\,ds.$$

$$(39)\qquad \lim_{x\to\infty} \frac{T(x - \epsilon(x - n)/T(n) + t)}{T(x - \epsilon(x - n)/T(n))} = 1.$$

and this limit holds uniformly for $t \in [0, 1]$, so that for $t = \epsilon(x - n)/T(n)$,

$$c_1(x) = \frac{T(x)}{T(x - \epsilon(x - n)/T(n))} \to 1 \quad \text{as } x \to \infty.$$

Therefore,

$$(40)\qquad T(x) = c_1(x)c(x)\exp\int_1^x \epsilon(s)\,ds,$$

and so $T \in K$.

Let $\{x_n\}$ be a sequence tending to infinity, and set

$$\{x_{n_\alpha}\} = \{x_n : \varphi(x_n) \leqslant 1\}, \quad \{x_{n_\beta}\} = \{x_n : \varphi(x_n) > 1\}.$$

Fix t, and set

$$\lambda_{n_\alpha} = tT(x_{n_\alpha}), \quad \mu_{n_\beta} = t + t\epsilon/\varphi(x_{n_\beta}).$$

Then $\lambda_{n_\alpha}, \mu_{n_\beta} \in [t, t + t\epsilon]$. Let $\eta > 0$ be given. Then there are integers N_1 and N_2 such that

$$(41) \qquad \begin{aligned} \left| \frac{T(x+\lambda)}{T(x)} - 1 \right| &< \eta \qquad (x \geqslant x_{N_1}; \lambda \in [t, t+\epsilon]), \\ \left| \frac{\varphi(x+\mu\varphi(x))}{\varphi(x)} - 1 \right| &< \eta \qquad (x \geqslant x_{N_2}; \mu \in [t, t+t\epsilon]), \end{aligned}$$

since $T \in K$ and $\varphi \in B_u$. Let $N = \max\{N_1, N_2\}$. Then, for $n \geqslant N$, $\varphi(x_n) \geqslant 1$,

$$(42) \qquad \begin{aligned} \left| \frac{T(x_n + tT(x_n))}{T(x_n)} - 1 \right| &= \left| \frac{\varphi(x_n + \mu_n \varphi(x_n)) + \epsilon}{\varphi(x_n) + \epsilon} - 1 \right| \\ &< \eta\varphi(x_n)/(\varphi(x_n) + \epsilon) < \eta. \end{aligned}$$

While if $\varphi(x_n) < 1$,

$$(43) \qquad \left| \frac{T(x_n + tT(x_n))}{T(x_n)} - 1 \right| = \left| \frac{T(x_n + \lambda_n)}{T(x_n)} - 1 \right| < \eta.$$

The bound in (41) holds uniformly for t in finite intervals, so $T \in B_u$, which completes the proof.

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