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AN ABELIAN ERGODIC THEOREM FOR SEMIGROUPS IN L_p SPACE

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ABSTRACT. The purpose of this paper is to prove individual and dominated ergodic theorems for Abel means of semigroups of positive L_p contractions, $1 < p < \infty$.

1. **Introduction.** Let (X, Σ, μ) be a σ-finite measure space and $L_p(\mu) = L_p(X, \Sigma, \mu)$, $1 \leq p \leq \infty$, the usual Banach spaces of complex-valued functions. Let $\{T(t)\colon t \geq 0\}$ be a strongly measurable semigroup of positive $L_p(\mu)$ contractions for some $1 < p < \infty$. This means that (i) $\|T(t)\|_p \leq 1$, $t \geq 0$; (ii) $0 \leq f \in L_p(\mu) \Rightarrow T(t)f \geq 0$; (iii) $T(s+t) = T(s)T(t)$, $s, t \geq 0$; (iv) $f \in L_p(\mu) \Rightarrow T(\cdot)f$ is measurable with respect to Lebesgue measure on the interval $[0, \infty)$. For $\lambda > 0$ we set

$$R_\lambda f(x) = \int_0^\infty e^{-\lambda t} T(t) f(x)\, dt$$

for $f \in L_p(\mu)$. In this paper we prove that

$$\int \Big(\sup_{\lambda > 0} |\lambda R_\lambda f(x)|\Big)^p d\mu \leq (p/p-1)^p \int |f|^p d\mu$$

and $\lim_{\lambda \to 0+} \lambda R_\lambda f(x)$ exists and is finite for a.e. $x \in X$. Before proceeding we justify the definition of $R_\lambda f(x)$. By Theorem III.11.17 in [3], given $f \in L_p(\mu)$ and $\lambda > 0$ the strong measurability of $\{T(t)\}$ guarantees the existence of a function $g_\lambda(t, x)$ on the product space $[0, \infty) \times X$, measurable with respect to the usual product σ-field, which is uniquely determined up to a set of measure zero in this space by the conditions (i) $g_\lambda(t, \cdot) = e^{-\lambda t} T(t) f$ for a.e. t, (ii) for a.e. x, $g_\lambda(\cdot, x)$ is integrable over $[0, \infty)$ and $\int_0^\infty g_\lambda(t, x)\, dt$ as a function of x is equal a.e. to $\int_0^\infty e^{-\lambda t} T(t) f\, dt$ defined as the L_p limit of Riemann sums. The set on which

$$\int_0^\infty g_\lambda(t, x)\, dt \neq \int_0^\infty e^{-\lambda t} T(t) f\, dt$$

is independent of $\lambda > 0$. We define $R_\lambda f(x) = \int_0^\infty g_\lambda(t, x)\, dt$. This justifies the definition of $R_\lambda f(x)$.

In a recent paper of R. Sato [7] it was shown that if $f \in L_p(\mu)$ then $\|f^*\| \leq (p/(p-1))\|f\|$ and $\lim_{\lambda \to 0+} \lambda R_\lambda f(x)$ exists and is finite a.e. on X. The function f^* is given by $f^* = \sup_{\lambda > 0} |\lambda R_\lambda f(x)|$. He also obtained a weak

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estimate for f^* in case $f \in L_1(\mu)$. Sato obtained these results assuming $\{T(t)\}$ to be a strongly measurable semigroup of (not necessarily positive) $L_1(\mu)$ contractions satisfying $\|T(t)f\|_\infty \leq \|f\|_\infty$ for all $f \in L_1(\mu) \cap L_\infty(\mu)$. In this paper we obtain Sato's L_p results assuming $\{T(t)\}$ is a semigroup of positive $L_p(\mu)$ contractions for some $1 < p < \infty$.

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2. **Preliminary results.** Our purpose in this section is to establish the dominated estimate $\|f^*\| \leq (p/(p-1))\|f\|$ for a discrete semigroup. Let T be a positive contraction of $L_p(\mu)$. Throughout this section we let $R_\lambda f = \sum_0^\infty \lambda^n T^n f$, $f \in L_p(\mu)$, $0 < \lambda < 1$, and $f^* = \sup_{0<\lambda<1} |(1-\lambda)R_\lambda f|$. We say that T admits of a dominated estimate with constant $c > 0$ if $\|f^*\| \leq c\|f\|$, $f \in L_p(\mu)$.

1. LEMMA. *Let T_n, $n = 1, 2, \ldots$, and T be positive contractions of $L_p(\mu)$ such that each T_n admits of a dominated estimate with constant c. If $\{T_n\}$ converges strongly to T then T also admits of a dominated estimate with constant c.*

PROOF. The argument is analogous to that appearing in [5, p. 369]. Let $A_1, \ldots, A_n$ be disjoint measurable sets and k a positive integer. For any $f \in L_p^+(\mu)$ and $0 < \lambda_j < 1$, $j = 1, 2, \ldots, n$, we have

$$\left\| \sum_{j=1}^n (1-\lambda_j)\chi_{A_j}(f + \lambda_j T_i f + \cdots \lambda_j^k T_i^k f) \right\| \leq c\|f\|$$

for $i = 1, 2, 3, \ldots$. Since $\{T_i\}$ converges strongly to T, the above estimate holds with T_i replaced by T. It follows that $\|\sum_j (1-\lambda_j)\chi_{A_j} R_{\lambda_j} f\| \leq c\|f\|$. By the monotone convergence theorem we get

$$\left\| \sup_{\substack{0<\lambda<1 \\ \lambda \text{ rational}}} (1-\lambda) R_\lambda f \right\| \leq c\|f\|.$$

Since $(1-\lambda)R_\lambda f$ depends continuously on λ, it follows that

$$\left\| \sup_{0<\lambda<1} (1-\lambda) R_\lambda f \right\| \leq c\|f\|,$$

$f \in L_p^+(\mu)$. Clearly the estimate also holds for arbitrary $f \in L_p(\mu)$. Q.E.D.

2. LEMMA. *Let (X, Σ, μ) be a Lebesgue space and T a positive invertible isometry of $L_p(\mu)$. Then T admits of a dominated estimate with constant $p/(p-1)$.*

PROOF. It is well known (see [5], [6]) that T is induced by an invertible point transformation and that, as a consequence of Linderholm's theorem [4, p. 71], T may be approximated in the strong operator topology by positive periodic isometries. Hence by Lemma 1 it is sufficient to prove the lemma assuming T is a positive periodic isometry. If $0 < f \in L_p(\mu)$ and T has period n, then $h = \sum_0^{n-1} T^i f$ is a positive fixed function for T, i.e. $Th = h$. As in [2] define a measure m on Σ by $m(A) = \int_A h^p \, d\mu$ and an operator P on $L_p(m)$ by $P(f) = T(fh)/h$, $f \in L_p(m)$. By Lemma 3.1 in [2], $\|P\|_1 \leq 1$, $\|P\|_\infty \leq 1$.

Consequently P admits of a dominated estimate with constant $p/(p-1)$ by Theorem 2 in [7]. For $f \in L_p(\mu)$, we have $f/h \in L_p(m)$ and $P^n(f/h) = T^n(f)/h$, $n = 0, 1, 2, \ldots$. Hence

$$\int \sup \left|(1-\lambda) \sum_0^\infty \lambda^n T^n f\right|^p d\mu = \int \sup \left|(1-\lambda) \sum_0^\infty \lambda^n P^n(f/h)\right|^p dm$$
$$\leq (p/(p-1))^p \int |f|^p d\mu.$$

Thus T admits of a dominated estimate with constant $p/(p-1)$. Q.E.D.

We now show that every positive contraction of $L_p(\mu)$ admits of a dominated estimate. We proceed as in [1]: the estimate is obtained first for positive contractions (matrices) operating on ℓ_p, where ℓ_p is the L_p space consisting of functions $r = (r_i) \in R_n$ whose norms are given by $\|r\|_p = [\sum_1^n |r_i|^p m_i]^{1/p}$, where the m_i's are fixed positive numbers. Some of the details in the proofs of the following lemmas are omitted since the arguments are similar to those in [1].

3. Lemma. *Let T: $\ell_p \to \ell_p$ be a positive contraction. Then T admits of a dominated estimate with constant $p/(p-1)$.*

Proof. The operator T is given by an $n \times n$ matrix (T_{ij}) whose entries T_{ij} are nonnegative. By Lemma 1 it is enough to establish the lemma assuming each $T_{ij} > 0$. Clearly we may assume $\|T\| = 1$. Given these conditions on T we construct as in [1] a space $(Z, \mathfrak{B}, \nu)$ where $Z = \bigcup_1^n E_i$, E_i a rectangle in R_2, $\mathfrak{B}$ is the collection of two dimensional Borel subsets of Z, ν is the restriction of two dimensional Lebesgue measure to $\mathfrak{B}$. The E_i's satisfy $\nu(E_i) = m_i$. For a given $r = (r_i) \in \ell_p^+$ set $f = \sum_1^n r_i \chi_{E_i}$. There exists a positive invertible isometry Q on $L_p(Z)$ such that for $i = 0, 1, 2, \ldots$

$$EQ^i f = \sum_{j=1}^n (T^i r)_j \chi_{E_j},$$

where E is the conditional expectation operator on $L_p(Z)$ with respect to $\{E_i\}$. Setting $f^* = \sup_{0<\lambda<1}(1-\lambda) \sum_0^\infty \lambda^i Q^i f$, we have

$$\|f^*\| \leq (p/(p-1))\|f\| = (p/(p-1))\|r\|$$

by Lemma 2. But $\sup_{0<\lambda<1}(1-\lambda) \sum_0^\infty \lambda^i EQ^i f \leq Ef^*$ and

$$\sup_{0<\lambda<1} (1-\lambda) \sum_0^\infty \lambda^i EQ^i f = \sup_{0<\lambda<1} (1-\lambda) \sum_{i=0}^\infty \sum_{j=1}^n \lambda^i (T^i r)_j \chi_{E_j}$$
$$= \sum_{j=1}^n r_j^* \chi_{E_j}.$$

Thus $\|r^*\| = \|\sum r_j^* \chi_{E_j}\| \leq \|Ef^*\| \leq (p/(p-1))\|r\|$. Q.E.D.

4. Lemma. *Let T be a positive contraction of $L_p(\mu)$. Then T admits of a dominated estimate with constant $p/(p-1)$.*

Proof. Suppose the theorem is false. Then there exists $f \in L_p^+(\mu)$, $K \geq 1$, $0 < \lambda_j < 1, j = 1, 2, \ldots, k$ such that

$$\left\|\sup_j (1-\lambda_j)\sum_{i=0}^{K}\lambda_j^i T^i f\right\| > (p/(p-1))\|f\|.$$

By Lemmas 3.1 and 3.2 in [1] there exists a conditional expectation E on $L_p(\mu)$ such that

$$\left\|\sup_j (1-\lambda_j)\sum_{i=0}^{K}\lambda_j^i (ET)^i Ef\right\| > (p/(p-1))\|Ef\|.$$

Let $\{E_1, \ldots, E_n\}$ be the partition of X corresponding to E and $\{E_{i_1}, \ldots, E_{i_m}\}$ the atoms of $\{E_i\}$ having finite positive measure. The subspace of $L_p(\mu)$ of functions which are constant on these atoms can be identified with ℓ_p and ET defines a positive contraction on this ℓ_p. Then the preceding inequality contradicts Lemma 3. Q.E.D.

3. **Main results.** Throughout this section we set

$$R_\lambda f(x) = \lambda\int_0^\infty e^{-\lambda t}T(t)f(x)\,dt$$

and

$$f^* = \sup_{0<\lambda<\infty}|\lambda R_\lambda f(x|, \qquad f\in L_p(\mu).$$

5. LEMMA. *For $f\in L_p(\mu)$ we have $f^*\in L_p(\mu)$ and*

$$\|f^*\| \le (p/(p-1))\|f\|.$$

PROOF. As in [7, pp. 544-545], one can show there exists a sequence $\{n_i\}$ such that for any rational $\lambda > 0$,

$$\lambda R_\lambda f(x) = \lim_i (1-e^{-\lambda/n_i})\sum_{k=0}^{\infty} e^{-\lambda k/n_i}T(k/n_i)f(x) \quad \text{a.e.}$$

Setting

$$f_i^*(x) = \sup_{0<\lambda<\infty}(1-e^{-\lambda/n_i})\sum_{k=0}^{\infty} e^{-\lambda k/n_i}T(k/n_i)|f|(x),$$

we have $|\lambda R_\lambda f(x)| \le \lim_i\inf f_i^*(x)$ a.e. for any rational $\lambda > 0$. Since the mapping $\lambda \to \lambda R_\lambda f(x)$ is continuous for a.e. x, it follows that

$$\sup_{0<\lambda<\infty}|\lambda R_\lambda f(x)| = \sup_{\substack{\lambda>0\\ \lambda\ \text{rational}}}|\lambda R_\lambda f(x)| \quad \text{a.e.}$$

Thus

$$f^*(x) \le \liminf f_i^*(x) \quad \text{a.e.}$$

By Fatou's lemma and Lemma 5 we have

$$\|f^*\| \le (p/(p-1))\|f\|.$$

This completes the proof.

6. THEOREM. *For any $f \in L_p(\mu)$, the limit*

$$\lim_{\lambda \to 0+} \lambda R_\lambda f(x)$$

exists and is finite a.e.

PROOF. The argument is the same as in [7]. For $1 < p < \infty$, $L_p(\mu)$ is reflexive and thus the vector subspace of functions f of the form

$$f = h + \sum_{i=1}^{n} [I - T(t_i)]g_i,$$

where $T(t)h = h$ for all $t \geq 0$ is dense in $L_p(\mu)$ (see Corollary VIII. 7.2 in [3]). Since

$$\begin{aligned} \lambda \int_0^\infty e^{-\lambda t} T(t)[I - T(t_i)]g_i(x)\,dt &= \lambda e^{\lambda t_i} \int_0^{t_i} e^{-\lambda t} T(t) g_i(x)\,dt \\ &\quad + \lambda(1 - e^{\lambda t_i}) \int_0^\infty e^{-\lambda t} T(t) g_i(x)\,dt \quad \text{a.e.} \end{aligned}$$

for each i, and

$$\lim_{\lambda \to 0+} \lambda e^{\lambda t_i} \int_0^{t_i} e^{-\lambda t} T(t) g_i(x)\,dt = 0 \quad \text{a.e.}$$

for each i, it follows from Lemma 5 that

$$\lim_{\lambda \to 0+} \lambda \int_0^\infty e^{-\lambda t} T(t)[I - T(t_i)]g_i(x)\,dt = 0 \quad \text{a.e.}$$

for each i. Thus $\lim_{\lambda \to 0+} \lambda R_\lambda f(x)$ exists and is finite for any f in a dense subset of $L_p(\mu)$. Hence the Banach convergence theorem [3, Theorem IV.11.3] implies that $\lim_{\lambda \to 0+} \lambda R_\lambda f(x)$ exists and is finite a.e. for all $f \in L_p(\mu)$. Q.E.D.

We remark that if we set $\tilde{f} = \lim_{\lambda \to 0+} \lambda R_\lambda f(x)$, then it follows from Lemma 5 and the Lebesgue dominated convergence theorem [3, III.6.16] that $\tilde{f} \in L_p(\mu)$ and $\lambda R_\lambda f(x)$ converges to $\tilde{f}$ in norm as well as pointwise.

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