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# ON NEUMER'S THEOREM

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ABSTRACT. In this note we propose to show that Neumer's theorem on regressive functions is actually a topological fact, by formulating and proving an entirely topological statement of which Neumer's theorem is an immediate corollary.

Let $\rho$ be an ordinal with $\mathrm{cf}(\rho) > \omega$. A subset $S \subset \rho$ is called stationary in $\rho$ if it intersects every closed cofinal subset of $\rho$. Neumer's theorem is the statement, very important and frequently used in set theory, saying that if $f$ is a regressive function on a stationary subset $S$ of $\rho$, i.e. $f(\alpha) < \alpha$ holds whenever $\alpha \in S$, $\alpha \neq 0$, then there is a cofinal subset $C \subset S$ and an ordinal $\sigma < \rho$ such that $f(\alpha) < \sigma$ for all $\alpha \in C$ (cf. [1]).

Now let $R$ be an arbitrary topological space and assume that we are given a family $\mathfrak{F}$ of infinite, closed, countably compact[1] subsets of $R$ with the following property:

(i) For any two members $F_1, F_2 \in \mathfrak{F}$ there exists an $F \in \mathfrak{F}$ such that $F \subset F_1 \cap F_2$.

It is obvious that for a $\rho$ as above (taken with its order topology) the final segments $\rho \backslash \alpha$ for $\alpha < \rho$ form a system with these properties. This example motivates, of course, our definitions.

Now a subset $C \subset R$ is called $\mathfrak{F}$-cofinal if $C \cap F \neq \varnothing$ for all $F \in \mathfrak{F}$, and $S \subset R$ is called $\mathfrak{F}$-stationary if $S \cap C \neq \varnothing$ for all closed $\mathfrak{F}$-cofinal subsets $C$ of $R$.

If $M$ is an arbitrary subset of $R$, an $\mathfrak{F}$-regressive function on $M$ is a map $U$ with domain $M$ such that for each $p \in M$ the value $U_p$ of $U$ at $p$ is an open neighborhood of $p$ with the following property:

(ii) If $p \in M$, $F \in \mathfrak{F}$ and $p \notin F$ then $U_p \cap F = \varnothing$. It should be clear that for our motivating example this is really the same as ordinary regressive functions.

Now we are in a position to formulate the generalization of Neumer's theorem.

THEOREM. *Let $R$, $\mathfrak{F}$ be as above, $S \subset R$ be $\mathfrak{F}$-stationary and $U$ be an $\mathfrak{F}$-regressive function on $S$. Then there is an $\mathfrak{F}$-cofinal set $C \subset S$ such that $\bigcap\{U_p : p \in C\} \neq \varnothing$.*

PROOF. Consider the closed set $K = R \backslash \bigcup\{U_p : p \in S\}$. Then $K \cap S = \varnothing$, hence $K$ cannot be $\mathfrak{F}$-cofinal as $S$ is $\mathfrak{F}$-stationary. Therefore, there exists an

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[1] By countably compact we mean here the property that every infinite subset has an accumulation point.

$F \in \mathfrak{F}$ with $K \cap F = \emptyset$, i.e. $F \subset \bigcup\{U_p : p \in S\}$.

A set $M \subset F$ is said to be $U$-free if for all $p \in S$ we have $|M \cap U_p| \leqslant 1$, i.e. no two distinct members of $M$ belong to the same $U_p$ for $p \in S$. Now being $U$-free is obviously a finite property, hence, by the Teichmüller-Tukey lemma, there is an $M \subset F$ which is a maximal $U$-free set.

We claim that $M$ is finite. Indeed, let $M$ be infinite. Since $F$ is countably compact, $M$ has an accumulation point $q$ in $F$, hence, as $F$ is covered by $\bigcup\{U_p : p \in S\}$, there is a $p \in S$ with $q \in U_p$. But then $U_p \cap M$ is infinite, which shows that $M$ cannot be $U$-free.

Thus we have $M$ a maximal, finite $U$-free subset of $F$, hence for every $x \in F \setminus M$ there are $p_x \in S$ and $q_x \in M$ such that $x$ and $q_x$ both belong to $U_{p_x}$. It follows immediately from (i) that the union of finitely many non-$\mathfrak{F}$-cofinal sets is not $\mathfrak{F}$-cofinal, and obviously $F \setminus M$ is $\mathfrak{F}$-cofinal. Consequently there is an $\mathfrak{F}$-cofinal subset $G \subset F \setminus M$ and a point $q \in M$ such that $q_x = q$ for all $x \in G$.

We claim that $C = \{p_x : x \in G\}$ is also $\mathfrak{F}$-cofinal. Indeed let $F_1$ be an arbitrary member of $\mathfrak{F}$. We have $G \cap F_1 \neq \emptyset$ because $G$ is $\mathfrak{F}$-cofinal; let $x \in G \cap F_1$. Then $x \in U_{p_x}$ by the choice of $p_x$, hence $U_{p_x} \cap F_1 \neq \emptyset$, which in turn implies $p_x \in F_1$ by the definition of regressive functions, consequently $C \cap F_1 \neq \emptyset$. Moreover we also have $q \in \bigcap\{U_{p_x} : p_x \in C\}$, which completes the proof of the Theorem.

Remark. Fodor's theorem (cf. [2]) is an improvement upon Neumer's theorem for ordinals $\rho$ with $\operatorname{cf}(\rho) > \omega$ saying that if $f$ is regressive on a stationary $S \subset \rho$, then there is actually a stationary set $S' \subset S$ on which the values of $f$ remain bounded below $\rho$. It would be interesting to see whether Fodor's theorem is also valid in this topological setting, maybe under some additional assumptions.

## References

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