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# SEQUENTIAL ORDER AND SPACES $S_n$

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ABSTRACT. It is shown that the space $\psi^*$ is a sequential space of order 2 which does not contain a copy of $S_2$. This solves a problem of Franklin and J. R. Boone. It is shown that there is a sequential space of order 4 and not of sequential order 3 but which still does not contain $S_3$. It is shown that for strongly sequential, and also for countable spaces, the problem of Franklin and Boone has an affirmative answer. Some open problems are raised.

Sequential spaces have been studied by Arhangelskii [1], Franklin [2], [3] and Kannan [5], [6]. Fréchet spaces are sequential spaces of order 1. There are plenty of sequential spaces which are not Fréchet. The easiest such space is the space $S_2$ of Arens [7]. Many authors observed that a sequential, Hausdorff space which is not Fréchet contains a copy of $S_2$ in general. This has led to the conjecture that a Hausdorff, sequential space which is not Fréchet must contain a copy of $S_2$. In this paper we show that this is not true. However we show that if the space is further countable then the conjecture is true. We also prove that if $X$ is a sequential, non-Fréchet, Hausdorff space which is strongly sequential, then $X$ contains a copy of $S_2$. Given an integer $n > 0$, we construct a Hausdorff sequential space of sequential order $> n$ and not containing $S_{n+1}$. We denote the set of integers by $Z$ and the set of positive integers by $Z^+$. We follow [10] in general for topological concepts.

DEFINITION 1. A topological space with a base point is a topological space $X$ with an element $x_0$ chosen out of it. We write $(X, x_0)$ for the space $X$ with a base point $x_0$. Two topological spaces with base points $(X, x_0)$ and $(Y, y_0)$ are said to be isomorphic if there is a homeomorphism $f$ from $X$ onto $Y$ such that $f(x_0) = Y_0$. In this case we write $(X, x_0) \sim (Y, y_0)$.

DEFINITION 2. Let $(X, x_0)$ be a topological space with a base point. Let $Y$ be a topological space and let $A$ be a subset of $Y$. For each $y \in A$ choose a topological space with a base point $(X_y, x_y)$ so that the following hold:

(i) $(X_y, x_y) \sim (X, x_0)$ for all $y \in A$.

(ii) $X_y \cap X_s = \varnothing$ for all $s, y \in A$ so that $s \neq y$.

Let $W$ be the free union of the spaces $(X_y)_{y \in A}$. Let $B \subset W$ be the set $\{x_y \mid y \in A\}$. Let $f: B \to Y$ be the map $f(x_y) = y$ for all $y \in A$. Then the join of $W$ and $Y$ through $f$ is called the space obtained from $Y$ by attaching a copy of $X$ through $x_0$ at each point of $A$.

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NOTATION 3. We denote by $S = Z^+ \cup \{\infty\}$ the one point compactification of the discrete space $Z^+$ of all positive integers. We call $S$ the space of a convergent sequence and $\infty$ is called the point at infinity of $S$ or the limit point of $S$. The only nonisolated point of $S$ is $\infty$. Sometimes we will use $S$ itself to denote the topological space with base point $(S, \infty)$. The meaning of $S$ will be clear from the context and will not be confusing. More generally if $X$ is a locally compact, noncompact Hausdorff space and $Y = X \cup \{\infty\}$ is the one point compactification of $X$, where $\infty$ is the point at infinity of $Y$, then we may write only $Y$ to denote the space with base point $(Y, \infty)$. Again the meaning will be clear from the context.

DEFINITION 4. The space obtained from $S$ by attaching a copy of $S$ through its limit point at each isolated point of $S$ is called $S_2$. (This is also called the Arens' space.) Let $n$ be an integer $> 1$ and suppose that the space $S_n$ has been defined. The space obtained from $S_n$ by attaching a copy of $S$ through its limit point at each isolated point of $S_n$ is called $S_{n+1}$.

DEFINITION 5. Let $X$ be a topological space and $A \subset X$. We denote by $A'$ or $A^1$ the set of all limits of convergent sequences from $A$. Let $\alpha$ be an ordinal $> 1$ and suppose we have defined $A^\gamma$ for all ordinals $\gamma < \alpha$. Then we put $A^\alpha = (\bigcup_{\gamma<\alpha} A_\gamma)'$. $A^\alpha$ is called the $\alpha$th sequential derivative of $A$. The space $X$ is called sequential if given a set $B \subset X$ there is an ordinal number $\beta$ so that $B^\beta = \overline{B}$, where $\overline{B}$ is the topological closure of $B$. The space $X$ is called Fréchet if $B' = \overline{B}$ for all subsets $B \subset X$. If $X$ is sequential then the least ordinal $\gamma$ so that $B^\gamma = \overline{B}$ for all $B \subset X$ is called the sequential order of $X$. It is denoted as $\sigma(X)$. (See [1].)

DEFINITION 6. Two subsets $A, B \subset Z^+$ are said to be equivalent if $(A \cup B) - (A \cap B)$ is finite. We write $A \approx B$ in this case.

LEMMA 7. *There exists a family $\mathfrak{F}$ of subsets of $Z^+$ with the following properties*:

(i) $|\mathfrak{F}| = C$ (*the cardinality of the set of all real numbers*).
(ii) *If $A, B \in \mathfrak{F}$ then either $A = B$ or $A \not\approx B$.*
(iii) *If $A, B \in \mathfrak{F}$ and $A \neq B$ then $A \cap B$ is finite.*
(iv) *If $D \subset Z^+$ and $D \cap A$ is finite for all $A \in \mathfrak{F}$ such that $A \not\approx D$, then there is a set $C \in \mathfrak{F}$ such that $D \approx C$.*

PROOF. A complete proof of this lemma will take us too long. A proof of this can be found in [4] and [9].

DEFINITION 8. Let $\mathfrak{F}$ be a family of subsets of $Z^+$ as in Lemma 7. For each $A \in \mathfrak{F}$, let us consider a symbol $\omega_A$. Let $\Psi = \{\omega_A | A \in \mathfrak{F}\} \cup Z^+$. We assume that $\omega_A \neq \omega_B$ if $A, B \in \mathfrak{F}$ and $A \neq B$ and $\omega_A \notin Z^+$ for all $A \in \mathfrak{F}$. We make $\Psi$ a topological space as follows: We declare $\{n\}$ to be open for all $n \in Z^+$. Given $A \in \mathfrak{F}$, a n.h.d. base of $\omega_A$ consists of sets of the form $\{\omega_A\} \cup F$ where $F \subset A$ and $A - F$ is finite.

REMARK. The space $\Psi$, equivalence relation $\approx$ of Definition 6, and the collection $\mathfrak{F}$ are known earlier. The idea of sequential order is given in [1]. We have given them all here together only for the reader's convenience. Now we are ready to give our promised examples and theorems.

LEMMA 9. *Let $\Psi^* = \Psi \cup \{\infty\}$ be the one point compactification of $\Psi$. Let $A \subset Z^+$ be infinite. Then we have that either,* (i) *there is a finite collection $F_1, F_2, \ldots, F_n$ of subsets of $Z^+$ so that $\omega_{F_1}, \omega_{F_2}, \ldots, \omega_{F_n} \in \Psi$ and $\bigcup_{i=1}^n F_i - A$ is finite, or* (ii) *$A$ contains a disjoint collection $A_1, A_2, \ldots, A_n \cdots$, of infinite subsets, and there are subsets $F_1, F_2, \ldots$ belonging to $\mathfrak{F}$ so that $A_n \subset F_n$ for all $n = 1, 2, 3, \ldots$. If case* (i) *does not happen then $\infty$ is a cluster point of $A$. Thus $\infty \in \overline{A}$ if and only if there exists an infinity of $F_1, F_2, F_3, \ldots, F_n, \ldots \in \mathfrak{F}$ such that $F_n \cap A$ is infinite for all $n = 1, 2, 3, \ldots$.*

PROOF. Obvious.

THEOREM 10. *There is a compact Hausdorff sequential, non-Fréchet space which does not contain a subspace homeomorphic to $S_2$.*

PROOF. We shall show that the compact, Hausdorff space $\Psi^*$ of Lemma 9 is such a space. It is easy to see that $\Psi^*$ is a sequential space of sequential order 2. From Lemma 9, it follows that $\infty$ is a cluster point of $Z^+$ and that no sequence in $Z^+$ converges to $\infty$. So $\Psi^*$ is not a Fréchet space.

Now it is not obvious to see that $\Psi^*$ does not contain a subspace homeomorphic to $S_2$. Now assume that, if possible, $X$ contains a subspace $Y$ which is homeomorphic to $S_2$. Since $Y$ is not Fréchet, it does not satisfy the first axiom of countability in the subspace topology. Since the first axiom is satisfied at all points of $\Psi^*$ except at $\infty$, we have that $\infty \in Y$. Now it is clear that $Y$ must contain a countably infinite set of nonisolated points where the first axiom is satisfied. So there are countably many sets $A_1, A_2, \ldots, A_n, \ldots$ so that $A_n \subset Z^+$ for all $n = 1, 2, 3, \ldots$ and subsets $B_1, B_2, \ldots, B_n, \ldots$ of $Z^+$ so that $B_n \in \mathfrak{F}$ for all $n = 1, 2, 3, \ldots$; $B_n \supset A_n$; $B_n - A_n$ is finite; $A_n \subset Y$ and $\omega_{B_n} \in Y$ for all $n = 1, 2, 3, \ldots$. Now put $M = \{\infty\} \cup \{\omega_{B_n} | n = 1, 2, 3, \ldots\} \cup \bigcup_{n=1}^\infty A_n$. It is clear that $M$ is also a subspace of $\Psi^*$ which is homeomorphic to $S_2$. However, we will show that $M$ contains a subset of $B$ such that $\infty \in \overline{B}$ and $\omega_{B_n} \notin \overline{B}$ for all $n = 1, 2, 3, \ldots$ which will give us a contradiction to the fact that $M$ is homeomorphic to $S_2$. Let us take $A_n = \{a_{n1}, a_{n2}, \ldots, a_{nk}, \ldots | k = 1, 2, 3, \ldots\}$ where $n = 1, 2, 3, \ldots$ and $a_{nk} \in Z^+$ for all $n$, $k = 1, 2, 3, \ldots$. Let us write $M$ as in the diagram below:

$$
\begin{array}{ccccc}
a_{11} & a_{12} & a_{13} & \cdots & a_{1k} \cdots \\
a_{21} & a_{22} & a_{23} & \cdots & a_{2k} \cdots \\
\vdots & \vdots & \vdots & \cdots & \vdots \\
\omega_{B_1} & \omega_{B_2} & \omega_{B_3} & & \omega_{B_k} \cdots \infty
\end{array}
$$

Let us call a set of the form $\{\omega_{B_k}\} \cup \{a_{1k}, a_{2k}, \ldots\}$ a column, where $k = 1, 2, 3, \ldots$. We will produce the required subset $B$ by induction. For this purpose we put $C = \bigcup_{n=1}^\infty A_n$ and take $B_1$ to be any infinite subset of $C$ which meets each column in a finite set. Now suppose that $n$ is an integer $\geqslant 1$ and we have defined an infinite subset $B_n \subset C$ so that $B_n$ intersects each column in a finite set. If $\infty \in \overline{B}_n$, then we take $B_n$ to be the required set $B$. If not, we have by Lemma 9 that there is a finite collection $F_1, F_2, \ldots, F_k$ of subsets of $Z^+$ such that $F_i \in \mathfrak{F}$ for all $i = 1, 2, 3, \ldots, k$ and $\bigcup_{i=1}^k F_i - B_n$ is finite. Then $B_n \cap F_i$

is infinite for some $i = 1, 2, 3, \ldots, k$. Without loss of generality we take that $B_n \cap F_i$ is infinite for all $i = 1, 2, \ldots, k$. Now choose some infinite subset $D \subset C$ so that the following hold:

(i) The intersection of $D$ and the $k$th column is empty for all $k = 1, 2, \ldots, n$.

(ii) $D \subset (C - \bigcup_{i=1}^{k} F_i) - B_n$.

(iii) $D$ intersects each column in a finite set.

Now put $B_{n+1} = B_n \cup D$. Then we arrive at either a set $B_k \subset C$ which is infinite and has $\infty$ as its cluster point and which meets each column in a finite set, or an infinite ascending sequence $B_1 \subset B_2 \subset \cdots B_n \subset \cdots$ of subsets of $C$ with the following properties:

(i) $B_{n+1} - B_n$ is infinite for all $n = 1, 2, 3, \ldots$.

(ii) There exists a sequence $F_1, F_2, \ldots, F_n, \ldots$ of distinct elements of $\mathfrak{F}$ such that $F_n \cap B_n$ is infinite for all $n = 1, 2, 3, \ldots$.

(iii) $B_n$ intersects each column in a finite set for all $n = 1, 2, \ldots$.

(iv) $B_n$ and $B_m$ intersect the $j$th column in the same set for all $m, n, j = 1, 2, 3, \ldots$ so that $j \leqslant n < m$.

Now put $B = \bigcup_{n=1}^{\infty} B_n$. Then $B$ intersects each column in a finite set and has $\infty$ in its closure. Thus we have a contradiction and our theorem is proved.

REMARK 11. The space $\Psi^*$ which served as our counterexample is also a chain compact space in the sense of Mrowka, Rajagopalan and Soundararajan [8]. However, it is not strongly scattered (see [8]). We will see later on that the conjecture is true for strongly scattered spaces.

THEOREM 12. *Given an integer $n \geqslant 2$, there is a sequential space of sequential order $n$ but not containing a copy of $S_n$.*

PROOF. Theorem 10 gives such a space for $n = 2$. If $n = 3$ then let $Y_3$ be the space obtained from $\Psi^*$ by attaching a copy of the space $S$ of a convergent sequence at each isolated point of $\Psi^*$ through its limit point. Then $Y_3$ is sequential and is of sequential order 3. Let $M \subset Y_3$ be homeomorphic to $S_3$. If $H \subset M$ is the set of all isolated points of $M$ then $M - H$ should be homeomorphic to $S_2$ and $M - H$ should be contained in $\Psi^*$ which is not possible by Theorem 10. For $n > 3$ the required space is obtained by induction. If $Y_{n-1}$ is constructed already, then $Y_n$ is obtained from $Y_{n-1}$ by attaching a copy of $S$ at each isolated point of $Y_{n-1}$ through its limit point. Then a similar argument as for $Y_3$ gives that $Y_n$ is a sequential space of sequential order $n$ but does not contain a copy of $S_n$.

REMARK. In Theorems 10 and 12 we got an example of a sequential space $X$ of sequential order '$n$' but not containing a copy of $S_n$. But we may conjecture that a space of sequential order $n$ should contain a copy of at least $S_{n-1}$. That even this weaker expectation does not hold is shown by

EXAMPLE 13. There exists a sequential space of sequential order 4 which does not contain a copy of $S_3$.

PROOF. Consider the space $\Psi_2^*$ obtained from $\Psi^*$ by attaching a copy of $\Psi^*$ at each isolated point of $\Psi^*$ through its point at infinity. Then $\Psi_2^*$ is sequential and of sequential order 4. However, $\Psi_2^*$ does not contain a copy of $S_3$. This follows by an argument similar to that used in Theorem 12.

REMARK 14. It is interesting to find out for what pairs $(n,k)$ of integers $n,k$

there is a sequential space $X$ of sequential order $n$ which does not contain a copy of $S_k$ with $k < n$. In particular, is there a sequential space $X$ of sequential order 3 but still not containing a copy of $S_2$?

So far we gave counterexamples to natural conjectures on sequential orders and $S_n$. Now we prove some positive theorems.

DEFINITION 15. Let $X$ be a sequential space. Then $X$ is called strongly sequential if $|A| = |A'|$ for all $A \subset X$.

THEOREM 16. *Let $X$ be a strongly sequential Hausdorff space which is not Fréchet. Then $X$ contains a copy of $S_2$.*

PROOF. Since $X$ is not Fréchet there exists a set $B \subset X$ so that $B' \neq B''$. Then there is an element $x_0 \in B'' - B'$ and a sequence $x_1, x_2, \ldots, x_n, \ldots$ in $B'$ so that $\mathrm{lt}_{n\to\infty} x_n = x_0$. Then there is a set $\{a_{mn} | m, n \in Z^+\}$ of distinct elements $a_{mn}$ in $B$ so that $\mathrm{lt}_{m\to\infty} a_{mn} = x_n$ for all $m,n = 1,2,3, \ldots$. Let $E = \{a_{mn} | m, n = 1,2,3,\ldots\} \cup \{x_n | n = 1,2,3,\ldots\} \cup \{x_0\}$. Let

$$A = \{a_{mn} | m, n = 1,2,\ldots\}.$$

Then no sequence in $A$ can converge to $a_0$. Now $A'$ is countable. Let $L = A' - E$. Suppose that $L$ is finite. Then there is open set $U$ of $X$ containing $x_0$ so that $L \cap \overline{U} = \varnothing$. Put $C = U \cap E$. Then $C$ is homeomorphic to $S_2$. For, the set of isolated points of $C$ is $C \cap A \subset U$. So, if $F \subset C \cap A$ and $x_n \notin F'$ for all $n = 1,2,3, \ldots$, then $F' = F$ and hence closed in $C$. Thus we see that $C$ is homeomorphic to $S_2$. Now let $L$ be infinite. Let $L = \{l_1, l_2, \ldots, l_n, \ldots\}$. Put $C_k = \{a_{mk} | m = 1,2,\ldots\}$ for all $k = 1, 2, 3, \ldots$. Let $V_n$ be a closed neighbourhood of $l_n$ so that $x_0 \notin V_n$ and $x_k \notin V_n$ for all $n = 1, 2, \ldots$ and $k = 1, 2, \ldots$. Then $V_n \cap C_k$ is finite for all $n, k = 1, 2, 3, \ldots$. Put $F_k = C_k - (C_k \cap \bigcup_{n=1}^{k} V_n)$ for all $k = 1, 2, 3, \ldots$. Finally put $M = \bigcup_{k=1}^{\infty} F_k \cup \{x_1, x_2, \ldots, x_n, \ldots\} \cup \{x_0\}$. Then $M \cap A$ is the set of all isolated points of $M$ and $x_n \in (M \cap A)'$ for all $n = 1,2,3, \ldots$. Now suppose that $T \subset M$ and $x_n \notin T'$ for $n = 1, 2, \ldots$. It is clear from the construction of $M$ that $l_n \notin T'$ for $n = 1, 2, \ldots$. Since $X$ is sequential, it follows that $T$ is closed. Thus $M$ is homeomorphic to $S_2$.

COROLLARY 17. *Let $X$ be a countable, Hausdorff sequential space which is not Fréchet. Then $X$ contains a copy of $S_2$.*

PROOF. The space $X$ is strongly sequential.

COROLLARY 18. *Let $X$ be a sequential, strongly scattered, compact Hausdorff space which is not Fréchet. Then $X$ contains a copy of $S_2$.*

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