

LARGE BASIS DIMENSION AND METRIZABILITY

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ABSTRACT. In this paper it is proved that if X is a regular Lindelöf space having finite large basis dimension, then X is metrizable if and only if it is a Σ -space or a $w\Delta$ -space.

Introduction. A collection Γ of subsets of a set X is said to have *rank 1* if whenever $g_1, g_2 \in \Gamma$ and $g_1 \cap g_2 \neq \emptyset$, then $g_1 \subset g_2$ or $g_2 \subset g_1$. According to P. J. Nyikos [8], a topological space X is said to have *large basis dimension* $\leq n$, denoted $\text{Bad } X \leq n$, if X has a basis which is the union of $\leq n + 1$ rank 1 collections of open sets. (A. V. Arhangel'skiĭ [1], [2], [3] uses the terminology "having a basis of big rank $\leq n + 1$ " instead of "large basis dimension $\leq n$ ".) $\text{Bad } X$ coincides with $\dim X$ and $\text{Ind } X$ for all metric spaces. For the case $n = 0$, the spaces become the nonarchimedean spaces of A. F. Monna [6].

It is the purpose of this paper to prove that every compact T_2 -space having finite large basis dimension is metrizable. This answers a question of Arhangel'skiĭ, first proposed in [2], where he proves that every compact nonarchimedean space is metrizable, and repeated in [3]. Some generalizations of the above result are also obtained.

Main result. All our spaces are assumed to be T_1 . The main result of this paper is the following:

THEOREM 1. *Let X be a regular Lindelöf space having finite large basis dimension. Then the following are equivalent:*

- (i) X is metrizable,
- (ii) X is a Σ -space [7],
- (iii) X is a $w\Delta$ -space [4].

As an immediate corollary, we have the result stated in the introduction:

COROLLARY 1. *Every compact T_2 -space having finite large basis dimension is metrizable.*

By [3, Lemma 3], if X has a basis $\Gamma = \bigcup \{\Gamma_i | i = 1, 2, \dots, n\}$ such that each Γ_i is a rank 1 collection, then $X = \bigcup \{X_i | i = 1, 2, \dots, n\}$ where X_i is such that Γ_i contains a local basis of each point of X_i . Our method of proving Theorem 1 is to show that if X is a regular Lindelöf space satisfying (ii) or (iii), then X is first countable and each X_i is Lindelöf. From this it will follow (see Lemma 3) that for each X_i there is a point-countable collection of open

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subsets of X which contains a basis of each point of X_i ; hence X has a point-countable basis, and is therefore metrizable by known results. It will be helpful to establish some lemmas.

LEMMA 1. *Let Γ be a rank 1 collection of open subsets of a space X which contains a basis at a point $x_0 \in X$. Let $\Gamma' \subset \Gamma$ and suppose $x_0 \in \cap \Gamma'$. Then either $x_0 \in \text{Int}(\cap \Gamma')$ or $\{x_0\} = \cap \Gamma'$. In the latter case, either Γ' contains a basis at x_0 , or x_0 is an isolated point.*

PROOF. Suppose there exists $y \in \cap \Gamma'$, $y \neq x_0$. Choose $g \in \Gamma$ such that $x_0 \in g$ but $y \notin g$. Then if $g' \in \Gamma'$, $g' \not\subset g$ and so $g \subset g'$. Hence $g \subset \cap \Gamma'$ and so $x_0 \in \text{Int}(\cap \Gamma')$.

Now suppose $\{x_0\} = \cap \Gamma'$, but that Γ' does not contain a basis at x_0 . Then there is some $g \in \Gamma$ which contains x_0 but does not contain any element of Γ' . Thus $g \subset g'$ for all $g' \in \Gamma'$, so $g \subset \cap \Gamma' = \{x_0\}$ and x_0 is therefore isolated.

Let Ω be the first uncountable ordinal, and let P_1 be the space obtained from the ordinal space $[0, \Omega]$ by isolating all ordinals less than Ω .

LEMMA 2. *Let X be a regular Lindelöf space having finite large basis dimension. Then either X is first countable, or X contains a closed subspace homeomorphic to P_1 .*

PROOF. Suppose X is not first countable at $x_0 \in X$. Let \mathcal{Q}_1 be an open cover of $X - \{x_0\}$ such that for each $U \in \mathcal{Q}_1$, $x_0 \notin \bar{U}$. By [3, Theorem 1], X is hereditarily metacompact. Let $\mathcal{V} = \{V_\alpha | \alpha \in A\}$ be a minimal point finite open refinement of \mathcal{Q}_1 . Since $\{x_0\} = \cap \{X - \bar{V}_\alpha | \alpha \in A\}$, \mathcal{V} must be uncountable, for otherwise it would follow from Lemma 1 that x_0 has a countable basis. Choose $x_\alpha \in V_\alpha - \cup \{V_\beta | \beta \neq \alpha\}$. Then $S' = \{x_\alpha | \alpha \in A\}$ has no cluster point in $X - \{x_0\}$. Since X is Lindelöf, every neighborhood of x_0 contains all but countably many elements of S' .

Let $S = S' \cup \{x_0\}$. We claim that S is homeomorphic to P_1 . To prove this, we need only show that if C is an infinite subset of S' such that $\text{card}(C) < \text{card}(S')$, then C is closed in S . To this end, let Γ be a rank 1 collection of open sets which contains a basis at x_0 , and for each $c \in C$ choose $U_c \in \Gamma$ such that $x_0 \in U_c$ but $c \notin U_c$. $U_C = \cap \{U_c | c \in C\}$ contains all but at most $\aleph_0 \cdot \text{card}(C) = \text{card}(C)$ elements of S' , and so by Lemma 1, $x_0 \in \text{Int}(U_C)$. Thus C is closed and the proof is finished.

LEMMA 3. *Let X be a first countable space, and let X' be a subspace of X such that some rank 1 collection Γ_0 of open subsets of X contains a basis at each point of X' . Suppose also that X' is Lindelöf. Then there exists a point-countable collection of open subsets of X which contains a basis at each point of X' .*

PROOF. Let \mathcal{C} be the set of all chains in Γ_0 (i.e., $C \in \mathcal{C}$ if C is a subset of Γ_0 and is totally ordered by inclusion), and let $\Gamma = \{\cup C | C \in \mathcal{C}\}$. By [9, Lemma 2], Γ has rank 1. Let $\Gamma' = \{g \cap X' | g \in \Gamma\}$. Clearly, Γ' is the set of all unions of chains in $\Gamma'_0 = \{g \cap X' | g \in \Gamma_0\}$. By [9, Lemma 2], the elements of Γ' are clopen (open and closed) subsets of X' . By [9, Theorems 3 and 4], X' can be partitioned into a collection \mathcal{Q}'_0 of disjoint elements of Γ' . Furthermore, we can ensure that this collection contains more than one element.

Clearly, any two elements of the corresponding collection \mathcal{U}_0 of elements of Γ are also disjoint.

We proceed to construct, for each $\alpha < \Omega$, a collection \mathcal{U}_α of disjoint elements of Γ . Suppose \mathcal{U}_α has been defined for all $\alpha < \beta$. Let $\mathcal{V}_\beta = \{\cap_{\alpha < \beta} U_\alpha \mid U_\alpha \in \mathcal{U}_\alpha\}$, and let $\mathcal{V}'_\beta = \{V \in \mathcal{V}_\beta \mid V \cap X' \text{ contains more than one point}\}$. By [3, Lemma 4], $V \cap X'$ is clopen in X' whenever $V \in \mathcal{V}'_\beta$. Thus $V \cap X'$ can be partitioned into a collection \mathcal{U}'_V of (more than one) disjoint elements of Γ' . Since $V \cap X' \subset \text{Int}(V)$, we can ensure that every element of the corresponding collection \mathcal{U}_V of elements of Γ is contained in V . Let $\mathcal{U}_\beta = \cup \{\mathcal{U}_V \mid V \in \mathcal{V}'_\beta\}$.

Suppose $V \in \mathcal{V}_\beta$, $V \cap X' = \{x_V\}$, and V contains more than one point of X . Then $x_V \in \text{Int}(V)$, and so there exists a local basis $\{g_n(V)\}_{n=1}^\infty$ of x_V such that $g_n(V) \subset V$ for all n . Let \mathcal{P}_β be the collection of all such $g_n(V)$'s.

Let $\mathcal{W} = \cup \{\mathcal{U}_\beta \cup \mathcal{P}_\beta \mid \beta < \Omega\}$. Since \mathcal{V}_β is a collection of disjoint sets, so is \mathcal{U}_β ; also, \mathcal{P}_β is point-countable, and $(\cup \mathcal{P}_\beta) \cap (\cup \mathcal{U}_\beta) = \emptyset$.

We claim that \mathcal{W} is point-countable. Choose $x_0 \in X$. If $x_0 \in \cup \mathcal{P}_\beta$, then $x_0 \notin \cup \{\mathcal{U}_\alpha \cup \mathcal{P}_\alpha \mid \alpha > \beta\}$. In this case, then, x_0 belongs to at most countably many elements of \mathcal{W} . Therefore, if x_0 is contained in uncountably many elements of \mathcal{W} , then for each $\alpha < \Omega$ there exists $U_\alpha \in \mathcal{U}_\alpha$ such that $x_0 \in U_\alpha$. By the way the U_α 's were constructed, if $\alpha < \alpha' < \Omega$, then $U_{\alpha'} \cap X' \subsetneq U_\alpha \cap X'$. Let $U_\Omega = \cap \{U_\alpha \mid \alpha < \Omega\}$. U_Ω cannot be clopen in X' , for otherwise $(X' \cap U_\Omega) \cup \{X' - U_\alpha \mid \alpha < \Omega\}$ is an open cover of X' with no countable subcover.

However, if $U_\Omega \cap X'$ is not clopen, then again by [3, Lemma 4], $U_\Omega \cap X' = \{x'\}$ for some $x' \in X'$. For each $\alpha < \Omega$, choose $x_\alpha \in (U_\alpha \cap X') - (U_{\alpha+1} \cap X')$. It is easy to see that x' is the only cluster point of $S = \{x_\alpha \mid \alpha < \Omega\}$ in X' . Since X' is Lindelöf, every neighborhood of x' must contain all but countably many elements of S , contradicting the fact that X is first countable. Therefore \mathcal{W} is point-countable as claimed.

Choose $x \in X'$. There exists a least ordinal β such that $x \notin \cup \mathcal{U}_\beta$. Let $\mathcal{U}_x = \{U_\alpha \mid x \in U_\alpha \in \mathcal{U}_\alpha, \alpha < \beta\}$, and let $\cap \mathcal{U}_x = V \in \mathcal{V}_\beta$. Then $V \cap X' = \{x\}$. Hence either $x \in g_n(V)$, $n = 1, 2, \dots$, or $V = \{x\}$, whence x is discrete in X or \mathcal{U}_x contains a local basis at x . Therefore $\mathcal{W} \cup \{x \in X' \mid x \text{ is discrete in } X\}$ is a point-countable collection of open subsets of X which contains a local basis at each point of X' , and the proof is finished.

PROOF OF THEOREM 1. The theorem is true if $\text{Bad } X = 0$ [8, Theorem 1.3]. Suppose it is true whenever $\text{Bad } X \leq k - 1$. Let X be a regular Lindelöf space with $\text{Bad } X \leq k$, i.e., X has a basis $\Gamma = \cup \{\Gamma_i \mid i = 1, 2, \dots, k + 1\}$ where each Γ_i has rank 1.

Since a paracompact $w\Delta$ -space is an M -space, and every M -space is a Σ -space, we need only prove that if X is a Σ -space, then X is metrizable. Since P_1 is not a Σ -space, by Lemma 2 X is first countable. Let X_i be the subspace of X such that $x \in X_i$ if and only if Γ_i contains a basis at x . We need only prove that X_i is Lindelöf, for then we can apply Lemma 3 to each X_i to show that X has a point-countable basis, from which it follows that X is metrizable [10].

X_1 is a nonarchimedean space, hence paracompact [9, Theorem 4]. Therefore if X_1 is not Lindelöf, there is an uncountable subset T of X_1 which has no

cluster point in X_1 . Consider the closure \bar{T} of T in X . The points of T are discrete in \bar{T} , so $\bigcup \{\Gamma_2 \cup T, \Gamma_3, \dots, \Gamma_{k+1}\}$ contains a basis (in the subspace \bar{T}) for each point of \bar{T} . Thus $\text{Bad } \bar{T} \leq k - 1$, so by the induction hypothesis, \bar{T} is metrizable. Thus $\bar{T} - T$ is G_δ in \bar{T} , and so there exists an uncountable subset T' of T which is closed in \bar{T} , and therefore in X , contradicting the fact that X is Lindelöf. Thus X_1 is Lindelöf. Similarly, X_2, \dots, X_{k+1} are Lindelöf, and the proof is finished.

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