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CENTER OF MASS AND *G*-LOCAL TRIVIALITY OF *G*-BUNDLES

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ABSTRACT. Riemannian geometry techniques are used to give a short and constructive proof that a differentiable G-fibre bundle with compact fibre is G-locally trivial when G is a compact Lie Group.

Introduction. The general notion of "*center of mass*" was introduced in [3] in order to give a constructive proof for the fact that any two C^1-close group actions of a compact Lie group G on a compact differentiable manifold M are equivalent. The information obtained from this constructive proof was important in proving an equivariant sphere-theorem (compare [4] and [5]). In addition, the "center of mass" was used in [4] and [5] to show that for any "*almost homomorphism*" between compact Lie groups there is a homomorphism close to it.

The purpose of the present note is to apply the "center of mass" to prove that any differentiable G-bundle with compact fibre is G-locally trivial when G is a compact Lie group. This was already proved in Bierstone [1]. However, besides being simpler, our proof has the advantage of being constructive. It is therefore likely to be useful in specific problems in, e.g., Riemannian geometry (compare [4] and [5]).

It is our hope that these applications (among others) will make it clear that the "center of mass" is useful as a technical tool in differential geometry as well as in differential topology. It can be considered as a nonlinear approach to standard linear averaging methods.

1. **The center of mass.** We recall some facts from [3] (see also H. Karcher [6]).

Let (X, μ) be a measure space with $\mu(X) = 1$ and let M be a Riemannian manifold. A measurable map $f: X \to M$ is said to be *almost constant* if its image $f(X)$ is contained in a sufficiently small convex ball B_ρ of radius ρ (depending on the curvature; see [3]). B_ρ is "sufficiently" small if the vectorfield

$$\mathcal{V}(q) = \int_X \exp_q^{-1} \circ f \, d\mu,$$

where $\exp_q: T_q M \to M$ is the exponential map, has a unique zero in B_ρ. This point is called the *center of mass* of f and it is denoted by $\mathcal{C}(f)$ (it is also characterized as the unique minimum for the convex function

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$$J(q) = \int_X d(f(\cdot), q)\, d\mu \text{ on } B_\rho,$$

where $d: M \times M \to \mathbf{R}$ is the distance function on M).

The important properties of $\mathcal{C}(f)$ are:

(1.1) *if* $\varphi: X \to X$ *is measure preserving, then* $\mathcal{C}(f \circ \varphi) = \mathcal{C}(f)$;

(1.2) *if* $A: M \to M$ *is an isometry, then* $\mathcal{C}(A \circ f) = A(\mathcal{C}(f))$.

Both properties are easy consequences of the construction of $\mathcal{C}(f)$.

Let now X be a compact topological space with borel measure μ. The Banach manifold $C^0(X, M)$ consisting of all continuous maps from X to M contains M as a submanifold (all the constant maps from X to M). In this case we can consider the set of "almost constant maps" $\mathfrak{A}$ as a tubular neighborhood of M in $C^0(X, M)$ and $\mathcal{C}: \mathfrak{A} \to M$ as a differentiable retraction of $\mathfrak{A}$ onto M (compare [3]).

2. G-**bundles.** *A differentiable G-fibre bundle* consists of a differentiable fibre bundle $\pi: E \to B$ together with a Lie group G acting differentiably on E and B such that π is equivariant. Two such G-bundles $\pi: E \to B$ and $\pi': E' \to B'$ are *G-isomorphic* if there is an equivariant bundle isomorphism between π and π'. Let $G_b = \{g \in G \mid g \cdot b = b\}$ denote the isotropy group of G at $b \in B$. Then G_b operates on the fibre $\pi^{-1}(b)$. $\pi: E \to B$ is said to be *G-locally trivial* if each point $b \in B$ has a G_b-invariant neighborhood U_b such that $\pi_l: \pi^{-1}(U_b) \to U_b$ is G_b-isomorphic to the trivial G_b-bundle $U_b \times \pi^{-1}(b) \to U_b$.

THEOREM 2.1. *Let* $\pi: E \to B$ *be a differentiable G-fibre bundle with compact fibre. If G is compact then* $\pi: E \to B$ *is G-locally trivial.*

PROOF. Since G is compact we can choose Riemannian metrics on E and on B such that G acts by isometries. Let $b_0 \in B$ and put $\pi^{-1}(b_0) = F$. Note that when we endow F with the induced metric from E then G_{b_0} operates on F by isometries. Obviously we may as well assume that $G_{b_0} = G$ and that $E = B \times F$. Denote this given action on $E = B \times F$ by $\mu_1: G \times B \times F \to B \times F$ and the product action of G on B and on F by $\mu_2: G \times B \times F \to B \times F$. We denote the actions of G on B and on F by μ_B and μ_F, respectively. Put $\Phi = P_F \circ \mu_1: G \times B \times F \to B \times F$, where $P_F: B \times F \to F$ is the projection onto F. Then Φ satisfies

$$\mu_F = \Phi| : G \times \{b_0\} \times F \to F, \tag{2.2}$$

$$\Phi(g_1 \cdot g_2, b, f) = \Phi(g_1, \mu_B(g_2, b), \Phi(g_2, b, f)). \tag{2.3}$$

Let now

$$\eta: G \times B \times F \to F; \quad (g, b, f) \to \Phi(g^{-1}, b_0, \Phi(g, b, f)),$$

and correspondingly,

$$\hat{\eta}: B \times F \to C^0(G, F); \quad (b, f) \to \eta(\cdot, b, f).$$

We can choose a G-invariant neighborhood of b_0, say U such that $\hat{\eta}(b)$ is "almost constant" for all $b \in U$. (Here we endow G with bi-invariant metric and corresponding measure of total volume 1.) Put $\psi = \mathcal{C} \circ \hat{\eta}_l \colon U \times F \to F$ and define $\Psi\colon U \times F \to U \times F$ by $\Psi(b,f) = (b, \psi(b,f))$ for all $(b,f) \in U \times F$. Then

$$\begin{aligned} \mu_2(g, \Psi(b,f)) &= (\mu_B(g,b), \mu_F(g, \psi(b,f))) \\ &= (\mu_B(g,b), \Phi(g, b_0, \mathcal{C}(\hat{\eta}(b,f)))) \quad \text{by (2.2)} \end{aligned}$$

and

$$\begin{aligned} \Psi(\mu_1(g,(b,f))) &= \Psi(\mu_B(g,b), \Phi(g,b,f)) \\ &= (\mu_B(g,b), \mathcal{C}(\hat{\eta}(\mu_B(g,b), \Phi(g,b,f)))). \end{aligned}$$

Now

$$\begin{aligned} \Phi(g, b_0, \mathcal{C}(\hat{\eta}(\ ,f))) &= \mathcal{C}(\Phi(g, b_0, \cdot) \circ \hat{\eta}(b,f)) && \text{by (1.2)} \\ &= \mathcal{C}(h \to \Phi(gh^{-1}, b_0, \Phi(h,b,f))) && \text{by (2.2)} \\ &= \mathcal{C}(h \to \Phi(h^{-1}, b_0, \Phi(hg,b,f))) && \text{by (1.1)} \\ &= \mathcal{C}(\hat{\eta}(\mu_B(g,b), \Phi(g,b,f))) && \text{by (2.3)}, \end{aligned}$$

i.e.

$$\mu_2(g, \Psi(b,f)) = \Psi(\mu_1(g,(b,f))) \quad \text{for all } (g,b,f) \in G \times U \times F.$$

Since $\psi(b,\cdot)\colon F \to F$ is C^1-close to 1_F when b is close to b_0, we can find a neighborhood $V \subset U$ of b_0 in B such that $\Psi\colon V \times F \to V \times F$ is a G-bundle isomorphism.

REMARK. Theorem 2.1 generalizes the result of Palais [7] and Calabi [2] about differentiable families of G-actions. Note also that the equivariant part of the pinching-theorem of [4], [5] can be formulated as follows. Let M be a complete Riemannian manifold whose sectional curvature satisfies $0 < \delta \leqq K \leqq 1$ and let G be a compact Lie group which operates on M by isometries. *There exists a* $\delta_0 < 1$ (independent of G and independent of dim M) *such that the* G*-bundle* $E = TM \oplus M \times \mathbf{R} \to M$ *is* G*-globally trivial when* $\delta > \delta_0$.

REFERENCES

1. E. Bierstone, *The equivariant covering homotopy property for differentiable G-fibre bundles*, J. Differential Geometry **8** (1973), 615–622.

2. E. Calabi, *On differentiable actions of compact Lie groups on compact manifolds*, Proc. Conf. on Transformation Groups (New Orleans, La., 1967), Springer, New York, 1968, pp. 210–213. MR **39** #6347.

3. K. Grove and H. Karcher, *How to conjugate C^l-close group actions*, Math. Z. **132** (1973), 11–20.

4. K. Grove, H. Karcher and E. Ruh, *Group actions and curvature*, Invent. Math. **23** (1974), 31–48.

5. ——, *Jacobi fields and Finsler metrics on compact Lie groups with an application to differentiable pinching problems*, Math. Ann. **211** (1974), 7–21.

6. H. Karcher, *Jacobifeld-Techniken in der Riemannschen Geometrie*, Lecture Notes in Math., Springer, Berlin and New York (to appear).

7. R. S. Palais, *Equivalence of nearby differentiable actions of a compact group*, Bull. Amer. Math. Soc. **67** (1961), 362–364. MR **24** #A185.

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