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PROCESSES WITH INFINITELY MANY JUMPING PARTICLES

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ABSTRACT. We give sufficient conditions for a Markov process of an infinite particle system to be specified by a formal generator which has a term for each finite subset of particles. Under stronger assumptions we show that processes of this type preserve a certain property of probability measures.

1. **Introduction.** In [1] Dobrushin initiated the study by probabilistic techniques of a class of models originating in nonequilibrium statistical mechanics. Within the model there are a countable infinity of particles labelled by the set S, and each particle is described by a point in its phase space W. When all the particles are fixed except the one labelled k, it undergoes a continuous time Markov jump process with specified generator G_k which depends on the configuration of all the particles. When the generator G_k is given for each $k \in S$, we have the *existence problem* of whether there is a Feller semigroup whose infinitesimal generator corresponds in a reasonable way to $\sum_{k \in S} G_k$.

Another type of infinite particle process was proposed by Spitzer [5], the existence problem being treated by Liggett [3]. In this model one has a countable number of indistinguishable particles moving on a set of sites S, each site being occupied by at most one particle. A configuration of the system is expressed by a point $x \in \{0, 1\}^S$, where the site k is occupied if $x_k = 1$, and otherwise unoccupied. A particle at k in the configuration x can jump to any of the unoccupied sites of x with infinitesimal transition probabilities depending on x and the pair of sites. An alternative way to view the model is to ascribe two states $\{0, 1\}$ to each point $k \in S$ and consider pairs of sites coherently changing states rather than particles jumping between sites. The model then takes the same form as that first described, except that one has a generator term for each pair of points of S, rather than for each point of S. A similar model with even more jump terms was considered by Holley [2].

The processes covered by the existence theorems of Dobrushin, Liggett and Holley exhibit two characteristic features. One is that, with probability 1, each particle undergoes at most a finite number of jumps in a finite time interval. The second is that the jumping behaviour of any one particle is controlled predominantly by finitely many other particles, in a certain sense.

In this paper we formulate an existence theorem for a process which allows coherent jumps for any finite subset of S, subject to conditions of the type mentioned in the previous paragraph. The result is based on ideas in [7] which

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also allow us to prove that a property of processes of this type discovered by Holley [2] extends with some additional assumptions to the case considered here.

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2. **Definitions and statement of existence theorem.** The one-particle phase space W is assumed to be compact and metrizable. The phase space of the system of particles $\Omega = W^S$ is compact and metrizable in the product topology. $C(\Omega)$ denotes the space of real valued continuous functions on Ω with the supremum norm $\|\cdot\|$. The term *measure* means bounded, countably additive Borel measure, and $\|\cdot\|_m$ denotes the total variation norm for measures. In the present context we employ measures parametrized by points of Ω, μ_x, and use the norm

$$\|\mu_x\| = \sup_{x \in \Omega} \|\mu_x\|_m. \tag{2.1}$$

The symbol Λ, possibly subscripted, always denotes a *finite* subset of S. The cardinality of Λ is denoted $|\Lambda|$. The limit $\lim \Lambda \to S$ is to be taken on the net of finite subsets of S ordered by inclusion. For $\Gamma \subset S$ we employ the left subscript notation

$$\begin{aligned} ({}_y x)_j &= y_j \quad \text{if } j \in \Gamma, \\ &= x_j \quad \text{if } j \notin \Gamma, \end{aligned} \tag{2.2}$$

for $x \in \Omega$, $y \in W^\Gamma$.

For $f \in C(\Omega)$ we define the sequence δf with values for each $j \in S$ as follows:

$$(\delta f)_j = \sup_{x, y \in \Omega; x = y \text{ except at } j} |f(x) - f(y)|, \tag{2.3}$$

$$\|\delta f\|_1 = \sum_{j \in S} (\delta f)_j. \tag{2.4}$$

DEFINITION. A *generator* G *on* Ω is a formal sum $\sum_\Lambda G_\Lambda$ with a term for each finite subset of S such that for each Λ and $x \in \Omega$, $G_\Lambda(x, \cdot)$ is a nonnegative measure on W^Λ which as a function of x is continuous in the topology of weak convergence of measures. The *operator* of G_Λ is the linear transformation of $C(\Omega)$ into itself given by

$$(G_\Lambda f)(x) = \int [f({}_y x) - f(x)] G_\Lambda(x, dy), \tag{2.5}$$

where the notation (2.2) is employed with $\Gamma = \Lambda$.

Given the generator $G = \sum G_\Lambda$ on Ω, we define $C(j, \Lambda)$ by

$$C(j, \Lambda) = \sup_{x, y \in \Omega; x = y \text{ except at } j} \|G_\Lambda(x, \cdot) - G_\Lambda(y, \cdot)\|_m. \tag{2.6}$$

This gives an estimate of the influence of the j-coordinate on G_Λ. Subtler estimates are employed in [7], but the above is sufficient for present considerations.

THEOREM 1. *Let* $G = \sum G_\Lambda$ *be a generator on* Ω *and* K *a real number such that*

$$\sum_{\Lambda \ni k} \|G_\Lambda\| \leqslant K, \tag{2.7}$$

$$\sum_{j \in S} \sum_{\Lambda \ni k} C(j, \Lambda) \leqslant K \tag{2.8}$$

for all $k \in S$ *with* $C(j, \Lambda)$ *given by* (2.6). *Then there is a strongly continuous, positive, linear semigroup of contractions* T_t, $t \geqslant 0$, *on* $C(\Omega)$ *so that for each* $f \in C(\Omega)$ *and each real* $t_0 > 0$,

$$\lim_{\Lambda \to S} \sup_{0 \leqslant t \leqslant t_0} \left\| T_t f - \exp\left(t \sum_{\Lambda' \subset \Lambda} G_{\Lambda'}\right) f \right\| = 0. \tag{2.9}$$

Further, if $\|\delta f\|_1 < \infty$, *then for each* $j \in S$,

$$(\delta T_t f)_j \leqslant (\exp(tC)\delta f)_j \tag{2.10}$$

where C *is the matrix with elements*

$$C_{jk} = \sum_{\Lambda \ni k} C(j, \Lambda). \tag{2.11}$$

The proof of Theorem 1 parallels, step by step, that of [7] and will be omitted.

3. **Approximate independence.** If the generator $G = \sum G_\Lambda$ on Ω has nonvanishing terms only for single point sets $\{k\}$ and $G_{\{k\}}(x, \cdot)$ depends only on x_k, then the associated semigroup T_t has the property that its adjoint T'_t (see [7]) maps product probability measures into product probability measures. Holley [2] observed that this is true in an approximate sense for the semigroups he considered. This section extends his result.

For $\Gamma \subset S$ we use the notation $C(\Omega|\Gamma)$ to denote those continuous functions which depend only on Γ-coordinates. A subset F of Ω is said to be Γ-*measurable* if it is measurable with respect to the smallest σ-field for which all functions of $C(\Omega|\Gamma)$ are measurable.

THEOREM 2. *Let the generator* $G = \sum G_\Lambda$ *on* Ω *satisfy the hypothesis of Theorem* 1 *and also satisfy*

$$\sum_{\Lambda} |\Lambda| C(k, \Lambda) \leqslant K, \tag{3.1}$$

$$\sum_{\Lambda \ni k} |\Lambda| \, \|G_\Lambda\| \leqslant K \tag{3.2}$$

for each $k \in S$. *Then for each* $f \in C(\Omega)$ *and each* $t_0 > 0$,

$$\lim_{\Lambda \to S} \sup_{g \in C(\Omega|\Lambda^c); \|g\| \leqslant 1} \sup_{0 \leqslant t \leqslant t_0} \|T_t(fg) - (T_t f)(T_t g)\| = 0. \tag{3.3}$$

Before proving the above we give two immediate corollaries. A probability measure μ on Ω is said to be *mixing* if for each Borel set E and each $\epsilon > 0$

there is a finite $\Gamma \subset S$ so that

$$|\mu(E \cap F) - \mu(E)\mu(F)| < \epsilon \tag{3.4}$$

for each Γ^c-measurable set F.

COROLLARY 1. *If μ is a mixing probability measure on Ω and T_t is the semigroup of Theorem 2, then $T_t'\mu$ is mixing for each real $t > 0$.*

When $S = Z^d$, the points with integer coordinates in d-dimensional Euclidean space, and G is translation invariant, one frequently considers probability measures which are *ergodic* in the sense of the Birkhoff theorem (see [2], [4], [6]).

COROLLARY 2. *Let $S = Z^d$ and let the G of Theorem 2 be translation invariant. If μ is a translation invariant and Birkhoff ergodic probability measure on Ω, then so is $T_t'\mu$ for each real $t > 0$.*

PROOF OF THEOREM 2. Select a fixed element $x^* \in \Omega$. For finite $\Gamma \subset S$ define the generator H on Ω as follows:

$$\begin{aligned} H_\Lambda(x, \cdot) &= G_\Lambda(y, \cdot) && \text{if } \Lambda \subset \Gamma \text{ with } y = x \text{ on } \Gamma, && y = x^* \text{ on } \Gamma^c, \\ &= G_\Lambda(z, \cdot) && \text{if } \Lambda \subset \Gamma^c \text{ with } z = x \text{ on } \Gamma^c, && z = x^* \text{ on } \Gamma, \\ &= 0 && \text{otherwise}. \end{aligned} \tag{3.5}$$

It follows that H satisfies the hypothesis of Theorem 1. Let U_t be the associated semigroup. Note for $f \in C(\Omega|\Gamma)$ and $g \in C(\Omega|\Gamma^c)$, $U_t(fg) = (U_t f)(U_t g)$. For $f \in C(\Omega)$ the difference between $T_t f$ and $U_t f$ can be estimated as follows:

$$T_t f - U_t f = \int_0^t \frac{d}{ds}(U_{t-s} T_s f)\, ds = \int_0^t U_{t-s}(G - H) T_s f\, ds, \tag{3.6}$$

$$\|T_t f - U_t f\| \leqslant \sum_{j,k \in S} h_j D_{jk} (\delta f)_k, \tag{3.7}$$

$$D_{jk} = \left[\int_0^t \exp(sC)\, ds\right]_{jk}, \tag{3.8}$$

$$h_j = \sum_{\Lambda \ni j} \|G_\Lambda - H_\Lambda\| \leqslant 2K. \tag{3.9}$$

The matrix C of (3.8) is given by (2.11). We note that D_{jk} of (3.8) is a nondecreasing function of t. We first obtain (3.7) under the assumption that $\|\delta f\|_1 < \infty$. The result extends by taking limits to all $f \in C(\Omega)$.

Because of (2.8) and (3.1) the D matrix satisfies

$$\sum_j D_{jk} \leqslant K^*, \qquad \sum_k D_{jk} \leqslant K^* = \int_0^t e^{sK}\, ds. \tag{3.10}$$

The h's can be estimated:

$$j \in \Gamma:\ h_j \leqslant \sum_{\Lambda \ni j; \Lambda \cap \Gamma^c \neq \varnothing} \|G_\Lambda\| + \sum_{k \in \Gamma^c} \sum_{\Lambda \ni j} C(k, \Lambda), \tag{3.11}$$

$$j \in \Gamma^c:\ h_j \leqslant \sum_{\Lambda \ni j; \Lambda \cap \Gamma \neq \varnothing} \|G_\Lambda\| + \sum_{k \in \Gamma} \sum_{\Lambda \ni j} C(k, \Lambda), \tag{3.12}$$

$$\sum_{j\in\Gamma^c} h_j \leqslant \sum_{\Lambda\cap\Gamma\neq\varnothing} |\Lambda|\,\|G_\Lambda\| + \sum_{k\in\Gamma}\sum_{\Lambda} |\Lambda| C(k,\Lambda). \tag{3.13}$$

We proceed to prove (3.3). It is sufficient to do so when f depends on only finitely many coordinates, i.e. $f \in C(\Omega|\Lambda_1)$. Given $\epsilon > 0$ and Λ_1 by (3.10) we can find $\Lambda_2 \supset \Lambda_1$ so that

$$\sum_{j\in\Lambda_2^c}\sum_{k\in\Lambda_1} D_{jk} \leqslant \epsilon. \tag{3.14}$$

From (3.11) and (2.7), (2.8) we can select $\Lambda_3 = \Gamma \supset \Lambda_2$, so that

$$\sum_{j\in\Lambda_2} h_j \leqslant \epsilon. \tag{3.15}$$

From (3.13) and (3.1), (3.2) we can select $\Lambda_4 \supset \Lambda_3$ so that

$$\sum_{j\in\Lambda_4^c} h_j \leqslant \epsilon. \tag{3.16}$$

Finally, from (3.10) we can find $\Lambda_5 \supset \Lambda_4$ so that

$$\sum_{j\in\Lambda_4}\sum_{k\in\Lambda_5^c} D_{jk} \leqslant \epsilon. \tag{3.17}$$

From these estimates we have for $f \in C(\Omega|\Lambda_1)$, $g \in C(\Omega|\Lambda_5^c)$:

$$\|T_t(fg) - U_t(fg)\| \leqslant 2\epsilon\|f\|\,\|g\|(K^* + 2K + 2K + K^*), \tag{3.18}$$

$$\|(T_tf)(T_tg) - U_t(fg)\| \leqslant 2\epsilon\|f\|\,\|g\|(2K^* + 4K). \tag{3.19}$$

The limit (3.3) follows.

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