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A REFINEMENT FOR COEFFICIENT ESTIMATES OF UNIVALENT FUNCTIONS

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ABSTRACT. By examining the coefficient inequalities of FitzGerald it is shown that if $f(z) = z + a_2 z^2 + a_3 z^3 + \ldots$ is analytic and univalent in the unit disc, then $|a_n| < (1.0691)n$.

Let

$$S = \left\{ f(z) = z + \sum_{n=2}^{\infty} a_n z^n : f \text{ is analytic and univalent in the unit disc} \right\}.$$

The famous conjecture of Bieberbach asserts that if $f(z) = z + a_2 z^2 + \cdots$ is in S, then $|a_n| \leqslant n$ $(n = 2, 3, 4, \cdots)$.

In [1] FitzGerald proves that

$$|a_n| < \sqrt{7/6}\, n < (1.0802)n \qquad (n = 2, 3, 4, \cdots) \tag{1}$$

and describes a method by which (1) can be improved. The purpose of this note is to carry out the first step in FitzGerald's program to obtain a refined coefficient bound for functions in the class S.

In [1] FitzGerald derives the following coefficient inequalities:

THEOREM 1 (FITZGERALD'S FIRST COEFFICIENT INEQUALITY). *If $f(z) = z + a_2 z^2 + a_3 z^3 + \cdots$ is in S, then*

$$|a_n|^4 \leqslant \sum_{k=1}^{n} k|a_k|^2 + \sum_{k=n+1}^{2n} (2n - k)|a_k|^2 \qquad (n = 2, 3, 4, \cdots). \square \tag{2}$$

THEOREM 2 (FITZGERALD'S SECOND COEFFICIENT INEQUALITY). *If $f(z) = z + a_2 z^2 + \cdots$ is in S, $\lambda_1, \lambda_2, \ldots, \lambda_L$ are complex numbers, and $n_1 \leqslant n_2 \leqslant \cdots \leqslant n_L$ are positive integers, then*

$$\begin{aligned} \left| \sum_{j=1}^{L} \lambda_j |a_{n_j}|^2 \right|^2 \leqslant{}& \sum_{j=1}^{L} |\lambda_j|^2 \left\{ \sum_{k=1}^{n_j} k|a_k|^2 + \sum_{k=n_j+1}^{2n_j} (2n_j - k)|a_k|^2 \right\} \\ & + 2\,\mathrm{Re} \sum_{1 \leqslant j_1 < j_2 \leqslant L} \lambda_{j_1} \bar{\lambda}_{j_2} \left\{ \sum_{k=n_{j_2}-n_{j_1}}^{n_{j_2}} (n_{j_1} - n_{j_2} + k)|a_k|^2 \right. \\ & \left. + \sum_{k=n_{j_2}+1}^{n_{j_1}+n_{j_2}} (n_{j_1} + n_{j_2} - k)|a_k|^2 \right\}. \square \end{aligned} \tag{3}$$

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FitzGerald derives the estimate (1) from (2). Note that (2) is a special case of (3), viz. $L = 1$.

Let n be any positive integer, and in (3) set $L = 2n$, $n_j = j$ for $1 \leqslant j \leqslant 2n$, and

$$\lambda_j = n - |n - j| \quad \text{for } j = 1, 2, \ldots, 2n. \tag{4}$$

The left-hand side of (3) then has the form of the right-hand side of (2), and therefore it follows that

$$\begin{aligned} |a_n|^8 \leqslant \sum_{j=1}^{2n} \lambda_j^2 &\left\{ \sum_{k=1}^{j} k|a_k|^2 + \sum_{k=j+1}^{2j} (2j - k)|a_k|^2 \right\} \\ &+ 2\sum_{l=2}^{2n} \sum_{m=1}^{l-1} \lambda_l \lambda_m \left\{ \sum_{k=l-m}^{l} (m - l + k)|a_k|^2 \right. \\ &\left. + \sum_{k=l+1}^{l+m} (m + l - k)|a_k|^2 \right\}. \end{aligned} \tag{5}$$

Let

$$C = \sup_n \sup_{f \in S} \left\{ \frac{|a_n|}{n} \right\}. \tag{6}$$

From (1) it follows that $C < \infty$. Given $\varepsilon > 0$ there exists a positive integer n and a function $f(z) = z + a_2 z^2 + a_3 z^3 + \cdots$ in S such that

$$n(C - \varepsilon) < |a_n|. \tag{7}$$

From (4), (5), (6), and (7) it follows that

$$\begin{aligned} n^8(C - \varepsilon)^8 \leqslant C^2 &\left[\sum_{j=1}^{2n} \lambda_j^2 \left\{ \sum_{k=1}^{j} k^3 + \sum_{k=j+1}^{2j} (2j - k)k^2 \right\} \right. \\ &+ 2\sum_{l=2}^{2n} \sum_{m=1}^{l-1} \lambda_l \lambda_m \left\{ \sum_{k=l-m}^{l} (m - l + k)k^2 \right. \\ &\left.\left. + \sum_{k=l+1}^{l+m} (m + l - k)k^2 \right\} \right]. \end{aligned} \tag{8}$$

Using the standard formulas for $\sum_{k=1}^{N} k^l$ for $l = 1, 2, 3, \ldots, 7$ one finds after a lengthy calculation that the right-hand side of (8) is equal to

$$C^2\left[\frac{1}{1260} (1881n^8 - 602n^6 + 49n^4 - 68n^2) \right] < C^2 \frac{1881}{1260} n^8.$$

Since $\varepsilon > 0$ was chosen arbitrarily, this implies

$$C^6 \leqslant 209/140 \quad \text{whence} \quad C \leqslant (209/140)^{1/6} < 1.0691.$$

This proves

THEOREM 3. *If $f(z) = z + a_2 z^2 + a_3 z^3 + \cdots$ is in S, then*

$$|a_n| \leqslant (209/140)^{1/6} n < (1.0691)n \qquad (n = 2, 3, 4, \cdots). \square \tag{9}$$

REMARK. FitzGerald suggests further slight refinements by substituting the right-hand side of (5) into the left-hand side of (3) and continuing this procedure of using (3) to bound the right-hand side of each successive inequality. However, any estimate that can be obtained on the right-hand side of (5) in this manner cannot be better than that which follows by replacing $|a_k|$ by k in this expression, since the Koebe function

$$f(z) = \frac{z}{(1-z)^2} = \sum_{k=1}^{\infty} kz^k$$

is in S. By doing so the inequality

$$n^8(C-\varepsilon)^8 \leqslant (209/140)n^8$$

is derived, from which it follows that the improved bounds described above can be no better than

$$C \leqslant (209/140)^{1/8} \cong 1.0514.$$

Thus, although the Bieberbach conjecture might still follow from inequality (3), it cannot be proven by successive applications of the above method.

REFERENCE

1. C. H. FitzGerald, *Quadratic inequalities and coefficient estimates for schlicht functions*, Arch. Rational Mech. Anal. **46**(1972), 356–368.

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